Thermally activated escape with potential fluctuations driven by an Ornstein-Uhlenbeck process

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We study the mean escape time $\bar{T}$ of an overdamped Brownian particle in a metastable potential that is subject to additive Gaussian white noise (thermal noise) and multiplicative Ornstein-Uhlenbeck noise (potential fluctuations). We derive two very general simple conditions for the existence of "resonant activation," i.e., a minimum of $\bar{T}$ as a function of the correlation time τ of the potential fluctuations. In the case of small thermal and potential fluctuations, we investigate $\bar{T}(\tau)$ for large τ by means of a kinetic model and the remaining τ regime by means of quasipotential theory. We find three different types of "resonant activation": a standard type, a type that typically occurs for potentials without fluctuations near the barrier and the well, and a mixed type.

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I. INTRODUCTION

The problem of thermally activated escape with *randomly* fluctuating potentials occurs in a wide variety of contexts [1–13]. Examples are molecular dissociation in strongly coupled chemical systems [14], oxygen binding to hemoglobin [15], models for the dynamics of a dye laser [16], selective pumps for biological macromolecules, chromosomes, or viruses [17], and recently introduced ratchet models for the action of molecular motors [18]. In a seminal paper, Doering and Gadoua [2] considered an overdamped Brownian particle driven by Gaussian white noise in a piecewise linear double well potential with slopes fluctuating according to a dichotomous process. They detected that the mean escape time $\bar{T}$ over the potential barrier may exhibit a minimum as a function of the correlation time τ of the potential fluctuations. This "astonishing phenomenon" [14], for which Doering and Gadoua coined the term *"resonant activation"* [19] stimulated numerous subsequent investigations: A more detailed study of the same model was carried out in Refs. [3,4]. Some light on the basic mechanism of resonant activation was shed by the investigation [5,6] of even much simpler kinetic models that, under appropriate conditions, exhibit the same behavior. More general potentials with fluctuations driven by an Ornstein-Uhlenbeck process were treated by means of different approximation schemes in Refs. [7,8] and, even before [2], in Refs. [9,10], while a simple particular model was solved exactly in [11]. The observation of resonant activation in analog simulations was reported in Ref. [12]. Finally, by means of a perturbation theory it could be shown [13] that the mean escape time $\bar{T}(\tau)$ generically decreases for sufficiently small correlation times τ and increases for asymptotically large correlations times τ, which implies the existence of resonant activation, for both dichotomous and Ornstein-Uhlenbeck noise driven potential fluctuations under the assumption of weak thermal noise.

In this article we continue our study [20] that predicted a variety of new general qualitative and quantitative properties of $\bar{T}(\tau)$ by means of simple intuitive arguments. Here, we present a more detailed and rigorous approach for the particular case where the potential fluctuations are driven by an Ornstein-Uhlenbeck process. We proceed as follows: In Sec. II we specify the model and introduce the central quantity of this paper, the mean escape time $\bar{T}(\tau)$. In Sec. III we determine $\bar{T}(\tau)$ for asymptotically small and large τ yielding as a by-product two very general simple conditions for the existence of resonant activation. In the rest of the paper we restrict ourselves to the most interesting case [21] of weak thermal and potential fluctuations. In Sec. IV we show that a kinetic model similar to those studied in Refs. [5,6] provides accurate approximations for $\bar{T}(\tau)$ in the large-τ regime. We review the well-known general properties of such kinetic models [22–24] and discuss the consequences with respect to our particular problem at hand. In Sec. V the rate concept [21] is adopted for the determination of $\bar{T}(\tau)$ in the τ regime not covered by the kinetic model. We restrict ourselves to the exponentially leading contributions for small thermal and potential fluctuations that can be determined by the powerful methods of quasipotential theory [25–34]. General qualitative properties of the quasipotential are discussed in Sec. VI, while a quantitative perturbation theory is presented in Sec. VII for potential fluctuations that are small in comparison with the thermal noise. The technically most involved calculations are carried out in Sec. VIII in order to determine the quasipotential for large correlation times τ. For readers who are not interested in the technical details *a rather complete summary of the main results is presented in Sec. IX.*

II. THE MODEL

We consider the usual one-dimensional model [2–4,7–13,20] of an overdamped Brownian particle with coordinate x,

$$\dot{x}(t) = -U'(x(t)) - W'(x(t))\, y(t) + \sqrt{2D}\; \xi(t), \quad (2.1)$$

additively disturbed by Gaussian noise $\xi(t)$ of vanishing mean and correlation $\langle\xi(t)\,\xi(s)\rangle = \delta(t-s)$ (thermal fluctuations). The particle moves in a potential

$$U_y(x) := U(x) + W(x)\,y \tag{2.2}$$

consisting of a static part $U(x)$ and a fluctuating part $W(x)$ that is driven by an Ornstein-Uhlenbeck process

$$\dot{y}(t) = -y(t)/\tau + \sqrt{2\,D/\tau}\;\eta(t)\ , \tag{2.3}$$

where $\eta(t)$ is the same kind of δ-correlated Gaussian noise as $\xi(t)$ in (2.1) with $\langle\xi(t)\,\eta(s)\rangle = 0$. The coupling strength $\sqrt{2\,D/\tau}$ of the white noise in (2.3) is chosen for later convenience. In principle, this does not yet mean a loss of generality since any other choice could be absorbed into the potential $W(x)$ in (2.1). We restrict ourselves to the case where the potential fluctuations (2.3) are in the *stationary* state. Thus their probability distribution $\rho(y)$ and correlation $C(t) := \langle y(t)\,y(0)\rangle$ are

$$\rho(y) = (2\,\pi\,D)^{-1/2}\ \exp\{-y^2/2D\} \tag{2.4}$$

and $C(t) = D\,e^{-|t|/\tau}$, respectively. The correlation time τ, $0 \le \tau \le \infty$, is the central control parameter of our model and we adopt the crucial assumption [2–4,11,13,20] that both the thermal noise and the distribution of the fluctuating potentials do not change with τ. In other words, we assume that $U(x)$, $W(x)$, and D in (2.1), (2.3) are τ *independent* [35].

For the sake of convenience only, we restrict ourselves to smooth metastable potentials $U(x)$ in (2.1) with a quadratic well (minimum) at $x = 0$ and a quadratic barrier (maximum) at $x = 1$. We further assume that $x = 0$ and $x = 1$ are the only zeros of $U'(x)$ and that $U(x)$ increases faster than what would be proportional to x^2 for large negative x [36], for instance,

$$U(x) = x^2/2 - x^3/3\ . \tag{2.5}$$

Finally, the fluctuating part of the potential $W(x)$ is assumed to be a smooth function that approaches a constant value for $x \to -\infty$ [37]. Two examples that we will often use in the following are

$$W(x) = \begin{cases} g\,U(x) & \text{for } x \ge 0 \\ g\,U(0) & \text{for } x \le 0 \end{cases} \tag{2.6}$$

and

$$W(x) = \begin{cases} g\,\cos^2(2\,\pi\,x) & \text{for } 1/4 \le x \le 3/4 \\ 0 & \text{otherwise}\ , \end{cases} \tag{2.7}$$

where g is a coupling constant. It turns out that the discontinuities of $W''(x)$ in these examples never impose a serious problem.

Our assumptions regarding $U(x)$ and $W(x)$ imply that the two-dimensional deterministic dynamics (2.1), (2.3) with $D = 0$ has a saddle point at $(x, y) = (1, 0)$ and a point attractor at the origin $(0, 0)$ with a τ-dependent basin of attraction G_τ. The basin boundary δG_τ cuts the x-y plane into two parts, one being deterministically attracted by the origin and the other by $x = \infty$, while on δG_τ itself, the saddle point $(1, 0)$ is the unique attractor.

We focus on an ensemble of particles (2.1) with an initial distribution $\rho_0(x)$ at time $t = 0$ that is mainly concentrated about the well $x = 0$ of the static potential $U(x)$. The quantity of central interest is the mean escape time $\bar{T}$ out of the region $x \le x_{\rm th}$, where the boundary $x_{\rm th}$ is required to be sufficiently far beyond the barrier $x = 1$ in order that particles, once they have crossed the threshold $x_{\rm th}$, are very unlikely to return into the region $x \le 1$ [38]. In particular, we will study the behavior of the mean escape time $\bar{T}(\tau)$ as a function of the correlation time τ of the potential fluctuations and the possible occurrence of resonant activation [nonmonotonic dependence of $\bar{T}(\tau)$ on τ with an absolute minimum at a finite τ value].

III. GENERAL FRAMEWORK

For small correlation times τ of the potential fluctuations (2.3) it is plausible that in leading order approximation their effect on the particle (2.1) is equal to that of white Gaussian noise (in Stratonovich interpretation [39]) of the same mean $\langle y(t)\rangle = 0$ and intensity $\int_{-\infty}^{\infty} C(t)\,dt = 2\,D\,\tau$. In other words, the fact that $y(t)$ is actually colored only concerns higher than leading order approximations in τ. These intuitive arguments are confirmed by the more detailed discussions in Refs. [8,10]. As a consequence, one readily finds [21] for the mean-first-passage time $T_\tau(x)$ across $x_{\rm th}$ for a particle with seed $x \le x_{\rm th}$ in leading order τ that

$$T_\tau(x) = \int_x^{x_{\rm th}} dv \int_{-\infty}^{v} dw\, \frac{\exp\left\{\int_w^v \frac{U'(z)}{D_\tau(z)}\,dz\right\}}{\sqrt{D_\tau(v)\,D_\tau(w)}}, \tag{3.1}$$

$$D_\tau(x) := D\,\{1 + \tau\,[W'(x)]^2\}. \tag{3.2}$$

On the other hand, in the limit $\tau \to \infty$, the potential fluctuations (2.3) become time independent. Consequently, by averaging over the mean-first-passage times for fixed $y(t)$ in (2.1) according to the probability distribution (2.4) one finds [21] that

$$T_\infty(x) = \int_x^{x_{\rm th}} dv \int_{-\infty}^{v} dw\, \frac{\exp\{F(v, w)/D\}}{D}, \tag{3.3}$$

$$F(v, w) := U(v) - U(w) + [W(v) - W(w)]^2/2. \tag{3.4}$$

From the mean-first-passage times (3.1) and (3.3) the corresponding mean escape times $\bar{T}(\tau)$ are obtained by averaging over the seeds,

$$\bar{T}(\tau) = \int_{-\infty}^{x_{\rm th}} \rho_0(x)\,T_\tau(x)\,. \tag{3.5}$$

From (3.1)–(3.5) we can infer that

$$\bar{T}(0) \le \bar{T}(\infty). \tag{3.6}$$

In fact, one even has the strict inequality $\bar{T}(0) < \bar{T}(\infty)$ except for a certain class of potentials $W(x)$ in the weak noise limit $D \to 0$, as we shall see in Sec. IV C. Furthermore, it can be shown that $\bar{T}(\infty)$ stays finite and $\bar{T}(\tau)$ is smooth and finite for small τ due to our assumptions at the end of Sec. II regarding the potentials $U(x)$ and $W(x)$. For a comparison of (3.1)–(3.5) with numerical simulations see Figs. 1–4.

According to (3.6), a sufficient condition for $\bar{T}(\tau)$ to exhibit a minimum (resonant activation) is that the asymptotic expression (3.1) decreases for increasing τ [20]. The latter is certainly the case if $W'(x) = 0$ whenever $U'(x) < 0$ and $x \le x_{\rm th}$. While *this simple sufficient condition for resonant activation is valid for general noise strengths D*, for small D one can evaluate (3.1) in saddle point approximation to yield for the mean escape time (3.5)

$$\bar{T}(\tau) = \bar{T}(0)\ \exp\left\{-\frac{\tau}{D}\int_0^1 U'(x)\,[W'(x)]^2\,dx\right\}\ , \quad (3.7)$$

$$\bar{T}(0) = \frac{2\,\pi}{|U''(0)\,U''(1)|^{1/2}}\exp\{\Delta U/D\}\ , \quad (3.8)$$

$$\Delta U := U(1) - U(0). \quad (3.9)$$

Thus, *for sufficiently small D resonant activation occurs generically* [13]; see also Figs. 1–4.

It is worth noting that in the weak-noise approximation (3.7) only the exponentially leading Arrhenius factor, but not the pre-exponential factor, depends linearly on τ. A similar weak-noise analysis as in (3.7)–(3.9) but for the large-τ limit (3.3), (3.5) is postponed to Sec. IV C.

In order to obtain quantitative results for correlation times τ more general than in (3.1)–(3.5), one cannot avoid dealing with the master equation

$$\dot{\rho}(x,y,t) = \Gamma\rho(x,y,t)\ ,\ \Gamma := \Gamma_x + \Gamma_y, \quad (3.10)$$

$$\Gamma_x := \partial_x[U'_y(x) + D\partial_x], \quad (3.11)$$

$$\Gamma_y := \partial_y[y + D\partial_y]/\tau, \quad (3.12)$$

governing the two-dimensional time dependent probability distribution $\rho(x,y,t)$ of the particle x and the potential fluctuations y. Here, the master operators Γ_x and Γ_y are the Fokker-Planck operators corresponding to the Langevin equations (2.1) and (2.3), respectively. The

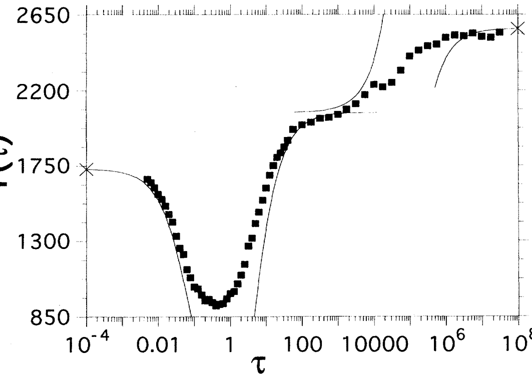

FIG. 2. Same as Fig. 1 but for the type II potential $W(x)$ from (2.7) with $g = 0.4$.

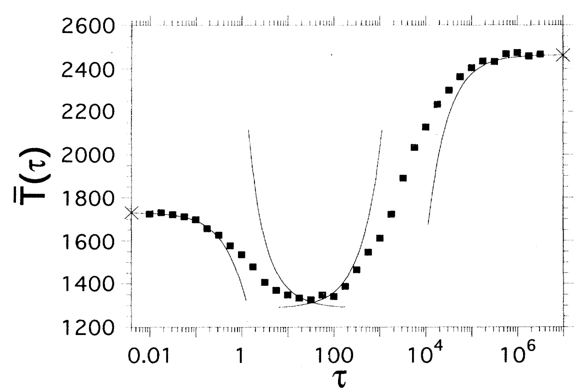

FIG. 1. Numerical simulations of the mean escape time $\bar{T}(\tau)$ out of the region $x \le x_{\rm th} = 3/2$ for the noisy dynamics (2.1), (2.3) with $U(x)$ from (2.5) and the type I potential $W(x)$ from (2.6) with $g = 1$. The noise strength is $D = 0.03$ and the initial distribution of particles $\rho_0(x) = \delta(x)$. The numerical uncertainty due to the time discretization and the finite number of realizations is a few percent. The crosses indicate the (exact) $\bar{T}(0)$ and $\bar{T}(\infty)$ from (3.1)–(3.5). The four solid lines represent the following theoretical approximations (from left to right): (1) small-τ asymptotics according to Eqs. (3.1), (3.5), (2) approximation (8.12) for $1 \ll \tau \ll \bar{T}_0$ with $\bar{T}_0$ from (4.7) and E from (8.11), (3) approximation (4.1), (4.6) for $\ln 1/D \ll \tau \ll \bar{T}(0)$ following from the kinetic model, and (4) large-τ asymptotics (4.1), (4.8) predicted by the kinetic model. Note that the predictions (2)–(4) are expected to become exact within the respective range of τ values only for asymptotically small D.

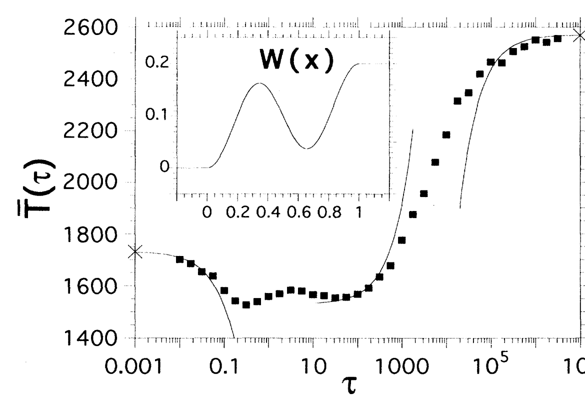

FIG. 3. Same as Fig. 1 but for the mixed type potential (shown in the inset) $W(x) = 0.025\,[1-\cos(\pi\,x)] + 0.075\,[1-\cos(3\,\pi\,x)]$ for $0 \le x \le 1$, $W(x) = 0$ for $x \le 0$, and $W(x) = 0.2$ for $x \ge 1$. The approximation (8.12) is not shown since the condition $U'(x) - \Delta W\,W'(x) > 0$ for $0 < x < 1$ is not satisfied.

evolution equation (3.10) is supplemented by the initial condition at time $t = 0$ reading $\rho(x,y,0) = \rho_0(x)\rho(y)$ and the boundary condition $\rho(x_{\rm th},y,t) = 0$ accounting for the absorption of particles after having escaped from the region $x \le x_{\rm th}$. The mean escape time then follows from $\rho(x,y,t)$ as

$$\bar{T}(\tau) = \int_{-\infty}^{x_{\rm th}} dx \int_{-\infty}^{\infty} dy \int_0^{\infty} dt\, t\,[-\dot{\rho}(x,y,t)] = \int_{-\infty}^{x_{\rm th}} dx \int_{-\infty}^{\infty} dy \int_0^{\infty} dt\, \rho(x,y,t) \ . \qquad (3.13)$$

In general, the solution of the master equation (3.10) is very difficult. *In the following we restrict ourselves to the most interesting case* [3,7–11,13,20,21] *of small strengths D of the thermal and potential fluctuations.* This case has the additional appealing property that the mean escape time $\bar{T}(\tau)$ becomes independent of the particular choice of the threshold $x_{\rm th}$, provided $x_{\rm th} > 1$, and the initial distribution $\rho_0(x)$ of particles, provided it is mainly concentrated about the potential well $x = 0$ [21].

Due to our specific choice $\sqrt{2D/\tau}$ of the coupling strength of the white noise $\eta(t)$ in (2.3) one should, in principle, allow for a D dependence of the potential $W(x)$. However, as already plausible by (3.1)–(3.5) and as will be confirmed by our subsequent calculations, the natural choice for small D is a (asymptotically) D-independent $W(x)$. Otherwise, either the thermal noise or the potential fluctuations in (2.1) becomes negligible for asymptotically small D [at least in the exponentially leading part of $\bar{T}(\tau)$]. We briefly come back to D-dependent potentials $W(x)$ in the final section IX.

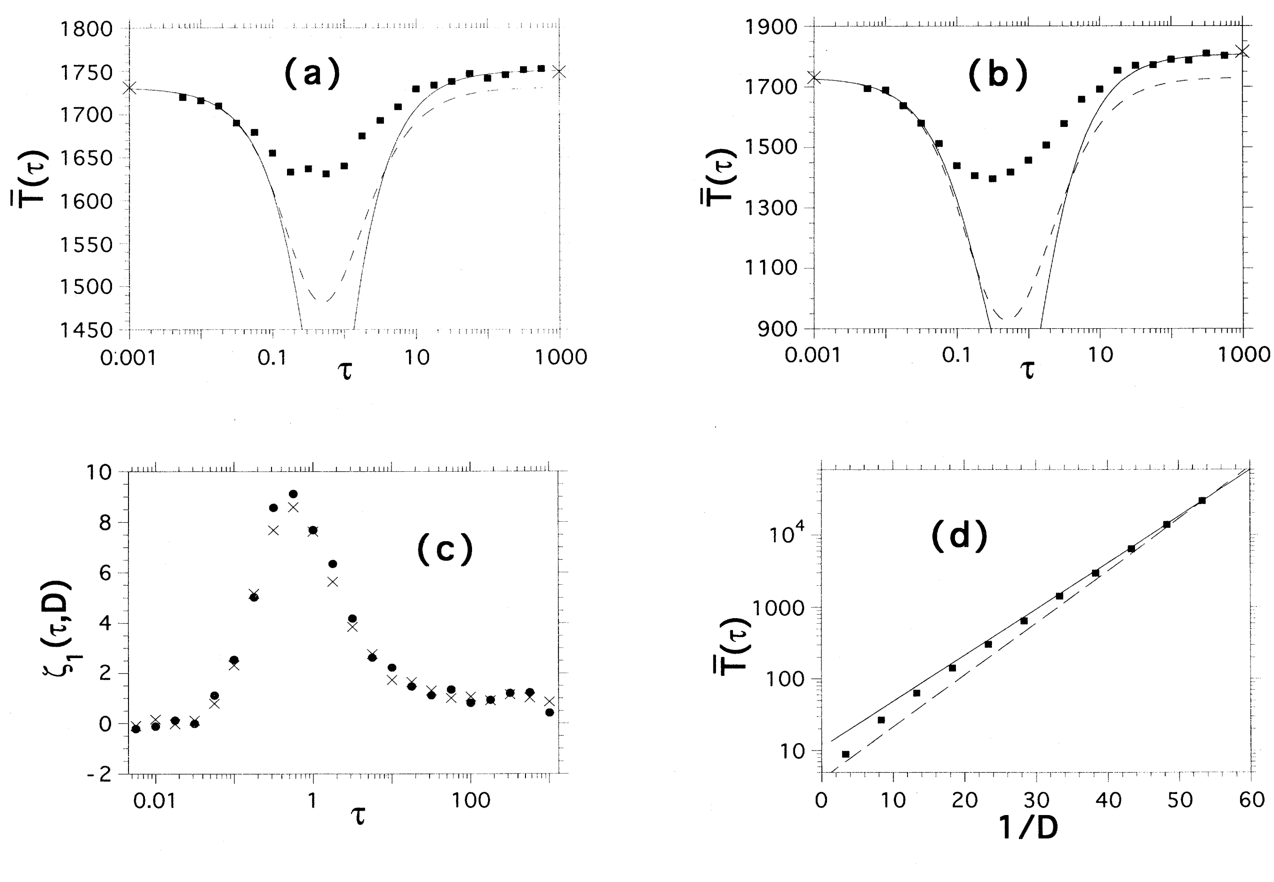

FIG. 4. Same as Fig. 1 but for the type II potential $W(x)$ from (2.7) with $g = 0.1$ (a) and $g = 0.2$ (b). The theoretical predictions (4.6), (4.8) are not shown since they practically coincide with $\bar{T}(\infty)$ within the range of their validity. The dashed line is the approximate theoretical prediction (7.7) with $\gamma = g$ and the exact $\bar{T}(0)$ from (3.1), (3.5). The qualitative agreement with the simulations is satisfactory, in particular regarding the characteristic features of $\bar{T}(\tau)$ in the type II case and the position of the minimal τ. (c) shows the unknown function $\zeta_1(\tau, D)$ from (7.8) for $g = \gamma = 0.1$ (dots) and $g = \gamma = 0.2$ (crosses) using the numerical results for $\bar{T}(\tau)$ and the theoretical predictions (3.1), (3.5), (7.6) for $\bar{T}(0)$ and $I(\tau)$. Within the numerical uncertainty $\zeta_1(\tau, D)$ is γ independent. (d) shows $\bar{T}(\tau)$ from numerical simulations at a fixed $\tau = 0.56234$ and $g = \gamma = 0.2$ for different noise strengths D. The solid straight line represents the theoretical approximation $\bar{T}(\tau) \simeq C\, e^{\Delta\phi(\tau)/D}$ with $\Delta\phi(\tau) = \Delta U - \gamma^2 I(\tau)$ according to (7.6). For a comparison $\Delta\phi(\tau) \equiv \Delta U$ is also shown as dashed line. In both cases, the constant C has been fitted to the numerical result at the largest $1/D$ value. The correctness of the solid straight line for asymptotically large $1/D$ is plausible.

IV. KINETIC MODELS FOR LARGE τ

A. Basic equations

For small strengths D of the thermal and potential fluctuations in (2.1), (2.3) it is plausible, and will be seen in more detail later on, that the mean escape time $\bar{T}(\tau)$ becomes large. If, additionally, the characteristic time τ of the potential fluctuations $y(t)$ is large then a particle typically spends most of this waiting time $\bar{T}(\tau)$ in a small neighborhood of the instantaneous absolute minimum $x_{\min}(y(t))$ of the "quasistatic" potential $U_{y(t)}(x)$ in the region $x \leq x_{\text{th}}$. The sojourn close to $x_{\min}(y(t))$ is interrupted by smaller and larger excursions that all end again near $x_{\min}(y(t))$ (unsuccessful escape attempts) and is terminated by a successful escape attempt. Hence, for correlation times τ much larger than the time scale T_a of the escape attempts, the particle sees in good approximation a nonfluctuating potential $U_y(x)$ during any excursion from the instantaneous minimum of the potential. In particular, for $\tau \gg T_a$ the probability per unit time to escape across x_{th} is given by the well-known Smoluchowski rate $k(y)$ corresponding to the instantaneous quasistatic potential $U_y(x)$ [21]

$$k(y) = \left[\int_{-\infty}^{x_{\text{th}}} dx \int_{-\infty}^{x} dz \times \frac{\exp\{[U_y(x) - U_y(z)]/D\}}{D}\right]^{-1} . \tag{4.1}$$

In other words, correlations between potential fluctuations and escape events are negligible [3]. Thus, for $\tau \gg T_a$ the joint probability $P(y,t) := \int_{-\infty}^{x_{\text{th}}} \rho(x,y,t)\, dx$ that a particle sees a quasistatic potential $U_y(x)$ and has not yet escaped from the region $x \leq x_{\text{th}}$ evolves under the simultaneous action of the master operator Γ_y governing the potential fluctuations y, see (2.3), (3.12), and the loss rate $k(y)$ due to successful escape events,

$$\dot{P}(y,t) = [\Gamma_y - k(y)]P(y,t). \tag{4.2}$$

The initial condition supplementing this so-called "kinetic equation" obviously reads $P(y,0) = \rho(y)$. By integration over t in Eq. (4.2) one finally finds that $P(y) := \int_0^\infty P(y,t)\, dt$ satisfies

$$[\Gamma_y - k(y)]P(y) = -\rho(y). \tag{4.3}$$

Alternatively, one may introduce a form function $f(y)$ through $f(y)\rho(y) := P(y)$ obeying

$$[\Gamma_y^\dagger - k(y)]f(y) = -1, \tag{4.4}$$

where $\Gamma_y^\dagger$ is the adjoint operator of Γ_y. Once (4.3) or (4.4) is solved, the mean escape time readily follows from (3.13) as

$$\bar{T}(\tau) = \int_{-\infty}^{\infty} P(y)\, dy = \int_{-\infty}^{\infty} \rho(y)\, f(y)\, dy . \tag{4.5}$$

For systems (2.1) with potential fluctuations driven by dichotomous noise $y(t)$, an approximative description analogous to (4.2) was introduced in Refs. [4,5] and investigated in further detail in Ref. [6]. Gain-loss balance equations of the form (4.2) have been studied extensively in the context of random walks with traps and the kinetics of diffusion-controlled chemical reactions, see [22–24] and further references therein. It is also worth mentioning that, apart from the absorption term $k(y)$ and the boundary conditions, Eq. (4.4) has the same form as the backward equations encountered in mean-first-passage-time problems [21].

Note that the kinetic model (4.1)–(4.4) itself is well defined for arbitrary D and τ. Unless stated otherwise, in the remainder of this section we mean by $\bar{T}(\tau)$ the hence following mean escape time (4.5). The latter is, however, expected to provide accurate approximations for the "true" mean escape time (3.13) of the particle (2.1) only for small D and large τ, as will be discussed in more detail in Sec. IV D. In particular, this will be always implicitly understood when comparing $\bar{T}(\tau)$ from (4.5) with the numerical results in Figs. 1–4.

B. General results

Obviously, there is little hope of solving (4.3) or (4.4) explicitly with $k(y)$ given by (4.1). However, asymptotic expressions for large and small τ are available [24], yielding for the mean escape time (4.5)

$$\bar{T}(\tau) = \bar{T}_0 + \frac{\tau}{D}\int_{-\infty}^{\infty} dy\, \frac{\left(\int_y^\infty d\tilde{y}\,[1 - \bar{T}_0\, k(\tilde{y})]\,\rho(\tilde{y})\right)^2}{\rho(y)} , \tag{4.6}$$

$$\bar{T}_0 := \left[\int_{-\infty}^{\infty} dy\, k(y)\, \rho(y)\right]^{-1} \tag{4.7}$$

for small τ and

$$\bar{T}(\tau) = \bar{T}(\infty) - \frac{D}{\tau}\int_{-\infty}^{\infty} dy\, \rho(y)\, [k'(y)]^2/[k(y)]^4 , \tag{4.8}$$

$$\bar{T}(\infty) = \int_{-\infty}^{\infty} dy\, \rho(y)/k(y) \tag{4.9}$$

for large τ. Furthermore, it can be proven [24] that $\bar{T}(\tau)$ *in* (4.5) *is strictly monotonically increasing with* τ *unless* $k(y)$ *is* y *independent in which case* $\bar{T}(\tau)$ *is constant.* In particular, it follows that $\bar{T}_0 \leq \bar{T}(\infty)$. For a comparison with numerical results see Figs. 1–4. The large-τ limit (4.9) easily follows from (4.3), (4.5) since Γ_y in (3.12) vanishes. On the other hand, for small τ the term $k(y)$ on the left-hand side of (4.3) becomes a small perturbation in comparison with Γ_y, suggesting an ansatz $P(y) \propto \rho(y)$. Introducing this ansatz into (4.3) and integrating both sides over y fixes the proportionality constant and with

(4.5) one recovers (4.7). The derivation of the leading order corrections (4.6), (4.8) of the asymptotic results (4.7), (4.9) as well as the proof that $\bar{T}(\tau)$ increases monotonically are more involved [24] and not reproduced here. We only mention that $\bar{T}_0 \leq \bar{T}(\infty)$ is an immediate consequence of the Cauchy-Schwartz inequality applied to (4.7) and (4.9). The fact that the asymptotically exact expressions (4.6) and (4.8) obviously overestimate and underestimate $\bar{T}(\tau)$ for sufficiently large and small τ, respectively, suggests that they are actually upper and lower bounds, in agreement with Figs. 1–3.

In order that the rate (4.1) correctly describe the decay from a "quasistatic" potential $U_y(x)$ we made the tacit assumption that this rate is equal to the inverse mean-first-passage time [21] across $x_{\rm th}$ for a particle (2.1) starting close to $x = 0$ and any y value that can occur with a nonnegligible probability $\rho(y)$. More precisely, this assumption just means that the expression (4.9) for $\bar{T}(\infty)$ agrees with the result (3.3)–(3.5) in the weak noise limit $D \to 0$. In other words, $F(v,w)$ in (3.4) is required to take its absolute minimum with side conditions $x \leq v \leq x_{\rm th}$ and $w \leq v$ at the same values v and w for any x with nonnegligible $\rho_0(x)$ as well as for $x = -\infty$. This imposes a (rather mild) condition on $W(x)$ in the region $x < 0$ additional to those introduced at the end of Sec. II.

C. Weak noise limit

In order to compare (4.6)–(4.9) with numerical simulations (for instance in Figs. 1–4) one preferably uses the full rate expression (4.1). However, for the further analytical discussion we restrict ourselves to the saddle point approximation

$$k(y) = \frac{|U_y''(a(y))\, U_y''(b(y))|^{1/2}}{2\pi} \times \exp\left\{\frac{U_y(a(y)) - U_y(b(y))}{D}\right\} \tag{4.10}$$

valid for asymptotically weak noise D, where $b(y)$ and $a(y)$ are the values of x and z, respectively, that maximize $U_y(x) - U_y(z)$ with the side condition $-\infty \leq z \leq x \leq x_{\rm th}$ [40]. With (4.10) one can rewrite (4.7) by means of another saddle point approximation under the form

$$\bar{T}_0 = \frac{2\pi\{1 + [W'(a)]^2/|U_{y_0}''(a)| + [W'(b)]^2/|U_{y_0}''(b)|\}^{1/2}}{|U_{y_0}''(a)\, U_{y_0}''(b)|^{1/2}} \exp\left\{\frac{U(b) - U(a) - [W(b) - W(a)]^2/2}{D}\right\} . \tag{4.11}$$

Here, the numerator in the exponential function is equal to the minimum of $y^2/2 + U_y(b(y)) - U_y(a(y))$, y_0 denotes the corresponding minimizing y, and $a := a(y_0)$, $b := b(y_0)$. In particular, one has that $y_0 = W(a) - W(b)$. Taking into account that $U_y(b(y)) - U_y(a(y))$ is nonnegative for any y and certainly positive for $y = 0$ it follows that $\bar{T}_0$ from (4.11) is of the form

$$\bar{T}_0 = C_1\, e^{C_2/D} \quad \text{with } C_1 > 0,\ C_2 > 0\ . \tag{4.12}$$

A more explicit evaluation of (4.11) is possible only under the assumption that $a(y = -\Delta W)$ and $b(y = -\Delta W) - 1$ are small quantities, where we introduced

$$\Delta W := W(1) - W(0). \tag{4.13}$$

It can be shown that this assumption is equivalent to the following two conditions: (i) α and β defined by

$$\alpha := \frac{\Delta W\, W'(1)}{|V''(1)|}\ ,\ \beta := \frac{\Delta W\, W'(0)}{|V''(0)|}\ , \tag{4.14}$$

$$V(x) := U(x) - \Delta W\, W(x) \tag{4.15}$$

are small quantities. (ii) The straightforward solution $x = 1 - \beta + O(\beta^2)$, $z = \alpha + O(\alpha^2)$ of $\partial_x[V(x) - V(z)] = 0$, and $\partial_z[V(x) - V(z)] = 0$ is not only an extremum of $V(x) - V(z)$ but the absolute maximum respecting the side condition $-\infty \leq z \leq x \leq x_{\rm th}$. Under these conditions (i) and (ii), it can be shown that the values of $a(y_0)$ and $b(y_0)$ that dominate the saddle point approximation (4.11) of (4.7), (4.10) are close to 1 and 0, respectively. As a consequence, one finally finds after some calculations that

$$\bar{T}_0 = \frac{2\pi\sqrt{C}}{|V''(0)\, V''(1)|^{1/2}} \exp\left\{\frac{\Delta U - \Delta W^2/2 + \kappa/2}{D}\right\}, \tag{4.16}$$

where contributions of order α and β in the prefactor and of third order in α and β in the exponential function are omitted and

$$C := 1 + [W'(0)]^2/|V''(0)| + [W'(1)]^2/|V''(1)| \tag{4.17}$$

$$\kappa := \frac{[\alpha^2|V''(0)| + \beta^2|V''(1)|]\,(2C - 1) - [\alpha\, W'(0) + \beta\, W'(1)]^2}{C^2}\ . \tag{4.18}$$

Similarly as in (4.11), a saddle point approximation of (4.1), (4.9) yields

$$\bar{T}(\infty) = \frac{2\pi\{1 - [W'(\tilde{a})]^2/|U''_{\tilde{y}_0}(\tilde{a})| - [W'(\tilde{b})]^2/|U''_{\tilde{y}_0}(\tilde{b})|\}^{-1/2}}{|U''_{\tilde{y}_0}(\tilde{a})\, U''_{\tilde{y}_0}(\tilde{b})|^{1/2}} \times \exp\left\{\frac{U(\tilde{b}) - U(\tilde{a}) + [W(\tilde{b}) - W(\tilde{a})]^2/2}{D}\right\}, \tag{4.19}$$

where $v = \tilde{b}$ and $w = \tilde{a}$ maximize $F(v,w)$ from (3.4) under the side condition $-\infty \leq w \leq v \leq x_{\rm th}$ and $\tilde{y}_0 := W(\tilde{b}) - W(\tilde{a})$. Furthermore, in the saddle point approximation (4.19) of (4.1), (4.9) the strict inequalities $-\infty < \tilde{a} < \tilde{b} < x_{\rm th}$ are assumed to hold. In order that (3.3)–(3.5) agree with (4.1), (4.9) in the weak-noise limit it is additionally required that $\tilde{b} > 0$; see also the last paragraph of Sec. IV B. The term raised to the power $-1/2$ in (4.19) is always positive as a consequence of the fact that $F(v,w)$ has a maximum at $v = \tilde{b}$, $w = \tilde{a}$. Similarly as in (4.16), under the additional conditions that $\tilde{a}$ and $1 - \tilde{b}$ are small quantities, Eq. (4.19) can finally be brought into the form

$$\bar{T}(\infty) = \frac{2\pi\sqrt{\tilde{C}}}{|\tilde{V}''(0)\,\tilde{V}''(1)|^{1/2}} \exp\left\{\frac{\Delta U + \Delta W^2/2 + \tilde{\kappa}/2}{D}\right\}, \tag{4.20}$$

where

$$\tilde{V}(x) := U(x) + \Delta W\, W(x), \tag{4.21}$$

$$\tilde{C} := \{1 - [W'(0)]^2/|\tilde{V}''(0)| - [W'(1)]^2/|\tilde{V}''(1)|\}^{-1}, \tag{4.22}$$

$$\tilde{\kappa} := \tilde{\alpha}^2|\tilde{V}''(0)| + \tilde{\beta}^2|\tilde{V}''(1)| + [\tilde{\alpha}\, W'(0) + \tilde{\beta}\, W'(1)]^2, \tag{4.23}$$

and $\tilde{\alpha}$, $\tilde{\beta}$ are as in (4.14) but with $\tilde{V}$ instead of V. Similarly as in (4.16), contributions of order $\tilde{\alpha}$ and $\tilde{\beta}$ in the prefactor and of third order in $\tilde{\alpha}$ and $\tilde{\beta}$ in the exponential function are omitted in (4.20). Clearly, these contributions vanish and (4.16), (4.20) become particularly simple if $\Delta W = 0$ or $W'(0) = W'(1) = 0$.

As mentioned in Sec. IV B, within the kinetic model $\bar{T}(\tau)$ becomes τ independent if $k(y)$ from (4.1) is y independent and increases strictly monotonically otherwise. The τ independence of $\bar{T}(\tau)$ is thus equivalent to $\bar{T}_0 = \bar{T}(\infty)$. From (4.1) it is clear that a constant $k(y)$ and thus $\bar{T}_0 = \bar{T}(\infty)$ is not possible for any finite D. However, within the validity of the saddle point approximation, i.e., for asymptotically small D, it can be concluded from (4.11) after some calculations that $\bar{T}_0 = \bar{T}(0)$ if

$$\Delta W = W'(0) = W'(1) = 0, \tag{4.24}$$

while $\bar{T}_0 < \bar{T}(0)$ if $\Delta W \neq 0$, and $\bar{T}_0 > \bar{T}(0)$ if $\Delta W = 0$ but $W'(0) \neq 0$ or $W'(1) \neq 0$. Here and in the following $\bar{T}(0)$ refers to (3.8), *not* to (4.6). Similarly, one can conclude from (4.19) that $\bar{T}(\infty) \geq \bar{T}(0)$, see also (3.6), and that the equality holds if and only if (4.24) and

$$\max_{-\infty \leq w \leq v \leq x_{\rm th}} F(v,w) = F(1,0) \tag{4.25}$$

are satisfied [41], where $F(v,w)$ is defined in (3.4). In summary, within the kinetic model $\bar{T}(\tau)$ is constantly equal to $\bar{T}(0)$ from (3.8) if D is asymptotically small and (4.24), (4.25) are satisfied and strictly monotonically increasing from $\bar{T}_0$ towards $\bar{T}(\infty) > \bar{T}(0)$ otherwise. For instance, the conditions (4.24), (4.25) are satisfied for the examples shown in Figs. 2 and 4 but not for those in Figs. 1 and 3. The deviations in Figs. 2 and 4 from $\bar{T}(\tau) = \bar{T}(0)$ for large τ are thus a finite-D effect. The fact that in Fig. 2 these deviations are quite considerable can be understood by noting that for a slightly larger g value than the one used in Fig. 2 (for instance $g = 0.41$) the condition (4.25) would no longer be fulfilled and thus $\bar{T}(\infty)$ would be exponentially larger than $\bar{T}(0)$ for weak noise D.

Regarding the leading order corrections (4.6) and (4.8) of the asymptotic behavior (4.7) and (4.9) we restrict ourselves to the case where we have

$$a(y) = 0\ ,\ b(y) = 1 \tag{4.26}$$

for all y values in (4.10) that notably contribute to the integrals in (4.6)–(4.9). In particular, this means that $W'(0) = W'(1) = 0$. Assuming (4.26), one readily recovers (4.16) and (4.20) with α, $\tilde{\alpha}$, β, $\tilde{\beta}$, κ, and $\tilde{\kappa}$ equal to 0. According to the derivation of (4.16) the assumption (4.26) is thus certainly justified as far as $\bar{T}_0$ is concerned if $V(x) - V(z)$ takes its absolute maximum respecting $-\infty \leq z \leq x \leq x_{\rm th}$ for $x = 1$ and $z = 0$. Similarly, the assumption (4.26) is certainly correct as far as $\bar{T}(\infty)$ is concerned if (4.25) and $W'(0) = W'(1) = 0$ are satisfied. Analogous conditions for the validity of the assumption (4.26) with respect to (4.6) and (4.7) are much harder to obtain. Here, we restrict ourselves to the following heuristic argument: The kinetic model (4.2) represents a random walk on the y axis with a y-dependent probability $k(y)$ per unit time that a random walker at y disappears from the system. It is plausible that the relative number of random walkers that disappear at any y for small τ differs only slightly from the case $\tau = 0$. In particular, the y values that dominate the mean escape time will be essentially the same. We thus can hope that the y values that notably contribute in (4.6) are the same as in (4.7). Analogous arguments for large τ suggest the same conclusion with respect to (4.8) and (4.9). In other words, the above-mentioned conditions that (4.26) is valid as far as (4.7) and (4.9) are concerned are expected to apply also for (4.6) and (4.8), respectively. In particular, these conditions are satisfied by the examples in Figs. 1–4 with the exception that for the one in Fig. 3, the maximum of $V(x) - V(z)$ is not at $x = 1$, $z = 0$.

Introducing (4.26) into (4.10) one obtains from (4.6)

$$\bar{T}(\tau) = \bar{T}_0[1 + \tau\, k(0)\, \psi(\Delta W/\sqrt{2D})] \tag{4.27}$$

$$\psi(z) := \frac{\sqrt{\pi}}{2} e^{-z^2} \int_{-\infty}^{\infty} dx\, e^{x^2} \left[\mathrm{erf}(x-z) - \mathrm{erf}(x)\right]^2 , \tag{4.28}$$

where $\mathrm{erf}(z) := 2\pi^{-1/2} \int_0^z e^{-x^2} dx$ is the error function. Obviously, we have $\psi(-z) = \psi(z)$ and, in particular, $\psi(0) = 0$. If z is of the order 1 or larger, then the integrand in (4.28) is approximately equal to $4\, e^{x^2}$ for $0 \le x \le z$ and negligible otherwise. Thus, $\psi(z)$ becomes roughly constant for large z. Finally, by means of (4.26), (4.10) we can rewrite (4.8) under the form

$$\bar{T}(\tau) = \bar{T}(\infty) \left(1 - \frac{1}{\tau\, k(0)} \frac{\Delta W^2}{D} e^{3\,\Delta W^2/2D}\right) . \tag{4.29}$$

For $\Delta W = 0$ one readily recovers from (4.27), (4.29) a constant $\bar{T}(\tau)$ as discussed below Eq. (4.25). For $\Delta W \neq 0$ we may consider (4.27) and (4.29) very roughly speaking, i.e., within exponential accuracy, as truncated series expansions in powers of $\tau/\bar{T}(0)$ and $\bar{T}(\infty)\, e^{\Delta W^2/D}/\tau$, respectively, where $\bar{T}(0)$ follows from (3.8) [*not* from (4.6) or (4.27)] and we used (4.10), (4.20), and (4.26). We thus expect that for $\Delta W \neq 0$ and weak noise D, Eqs. (4.6) and (4.8) will be accurate approximations at least for small $\tau/\bar{T}(0)$ and large $\bar{T}(\infty)\, e^{\Delta W^2/D}/\tau$, respectively, in good agreement with Figs. 1 and 3.

We finally mention that even under the conditions (4.26) and $W''(0) = W''(1) = 0$, implying for the rate (4.10) the simple form $k(y) = k(0)\, e^{-y\,\Delta W/D}$, a closed analytical solution of (4.3) or (4.4) is not known. For numerical solutions see Ref. [23].

D. Consistency

Formally, our intuitive derivation of (4.2) corresponds to an adiabatic elimination (Born-Oppenheimer approximation) of x in the master equation (3.10), i.e., to approximating the operator Γ_x by its smallest "instantaneous eigenvalue" $-k(y)$ and then integrating over x [13]. This is justified as long as both τ^{-1} and $k(y)$ are much smaller than the other eigenvalues of Γ_x describing the "intrawell relaxation" towards the instantaneous quasistationary state corresponding to the potential $U_y(x)$ [13]. The time scale of the intrawell relaxation can be readily identified with the time scale T_a of the escape attempts. Moreover, the assumption in Sec. IV A that $\bar{T}(\tau)$ and τ are large and the escape from any quasistatic potential $U_y(x)$ can be described by a rate (4.1) are equivalent to the conditions $\tau \gg T_a$ and $[k(y)]^{-1} \gg T_a$ in the above-mentioned adiabatic elimination of x. Strictly speaking, not only $k(y)$ but also T_a depends on y and the conditions $\tau \gg T_a$ and $[k(y)]^{-1} \gg T_a$ must be fulfilled for any real y since the Ornstein-Uhlenbeck noise (2.3) is unbounded. In the following we will see that there is hardly any potential $W(x)$ for which these rigorous conditions are actually satisfied. However, we will argue that the kinetic model is expected to provide accurate approximations for $\bar{T}(\tau)$ under much weaker conditions, basically since large y values that may violate $\tau \gg T_a$ or $[k(y)]^{-1} \gg T_a$ only occur with extremely small probability $\rho(y)$.

It has been shown in Ref. [33] that for a potential $U_y(x)$ with a single minimum in the domain $x < x_{\mathrm{th}}$, for instance $U_y(x) = U(x)$, the time scale of an escape attempt is given by

$$T_a \sim \ln 1/D \tag{4.30}$$

for small noise strengths D, where the proportionality constant depends on the details of $U_y(x)$. In view of (4.1), (4.10) it follows that the conditions $[k(y)]^{-1} \gg T_a$ and $\tau \gg T_a$ require small noise strengths D and $\tau \gg \ln 1/D$. We thus expect that weak noise D and $\tau \ge \tau_{\mathrm{min}}(D)$ with $\tau_{\mathrm{min}}(D) \gg \ln 1/D$ are necessary conditions for the kinetic model to provide good approximations for the mean escape time $\bar{T}(\tau)$. In particular, for $\tau \to \infty$ we have seen in the last paragraph of Sec. IV B that the correct $\bar{T}(\tau)$ is recovered from the kinetic model only if the rate (4.1) is equal to the inverse mean-first-passage time across x_{th} which generically is only true for small noise strengths [21].

We conjecture that *weak noise D and large correlation times $\tau \ge \tau_{\mathrm{min}}(D)$ are not only necessary but also sufficient* conditions for the kinetic model to provide accurate approximations for $\bar{T}(\tau)$ with $\tau_{\mathrm{min}}(D)$ diverging for $D \to 0$ much faster than $\ln 1/D$ but much slower than $\bar{T}_0$. In the following we give some evidence for this conjecture, however, without being able to prove it. We first mention the agreement with the numerical results in Figs. 1–4 apart from finite-D effects. Next we note that according to (4.12) the restrictions $\tau_{\mathrm{min}}(D) \gg \ln 1/D$ and $\tau_{\mathrm{min}}(D) \ll \bar{T}_0$ are compatible. Moreover, since $\bar{T}(\tau) \ge \bar{T}_0$ within the validity of the kinetic model, we are consistent with the assumption at the very beginning of Sec. IV A that $\bar{T}(\tau)$ becomes large for small D.

If the adiabatic elimination of x is not justified for a certain y value or, equivalently, a rate description fails for the corresponding potential $U_y(x)$, this induces two kinds of errors in the kinetic model in comparison with the true behavior of the particles (2.1). First, the escapes from such a potential $U_y(x)$ are not properly taken into account in the mean escape time. Second, the number of particles that do *not* escape until $U_y(x)$ notably fluctuates is generally wrong, undermining the validity of the kinetic model even for those y values for which a rate description is valid. Thus, if the rate description fails for a y value, we have to make sure that both kinds of errors are negligible.

From (4.30) one can readily conclude that a rate description is certainly valid for potentials $U_y(x)$ with a single minimum in the region $x < x_{\rm th}$ [weak noise D and $\tau \geq \tau_{\rm min}(D)$, $\tau_{\rm min}(D) \gg \ln 1/D$, is always tacitly assumed]. Next we consider potentials $W(x)$ and y values such that $U_y(x)$ is monotonically decreasing in the whole region $x \leq x_{\rm th}$. An example is (2.6) but with $W(x) = g\,U(1)$ for $x \geq 1$, $g > 0$, and $y < -1/g$. It is obvious that a rate description fails for such a potential $U_y(x)$. In particular, both the formal inverse escape rate (4.1), (4.11) and the true escape time will be much smaller than $\tau_{\rm min}(D)$. In other words, for such a potential $U_y(x)$ an escape takes place almost certainly until the potential notably fluctuates but the same would also be true if the "wrong" rate description were still used. Moreover, using the wrong rate description only leads to a negligible error in the mean escape time $\bar{T}(\tau)$ since $\tau_{\rm min}(D) \ll \bar{T}_0 \leq \bar{T}(\tau)$. In summary, our conjecture regarding the validity of the kinetic model [which is based on a rate description for all $U_y(x)$] is consistent even in the case where $U_y(x)$ exhibits no potential barrier for certain y values.

While our conjecture is thus rather suggestive in the case where $U_y(x)$ can only exhibit one or no minimum in the domain $x \leq x_{\rm th}$ for all real y, things become different if $U_y(x)$ exhibits more than one minimum for certain y values. For instance, $U_y(x)$ may display two minima that are separated by a very high barrier such that the time scale T_a of the "intrawell relaxation," which is, in this case, the transition time over the high barrier, becomes comparable or larger than τ or the (formal) inverse rate (4.1). Our conjecture expresses the hope that such y values only negligibly contribute to the number of escape events and the mean escape time $\bar{T}(\tau)$ both in the kinetic model and in the true escape problem for the particle (2.1).

V. RATE CONCEPT AND QUASIPOTENTIALS

For small noise strengths D one expects a rather extended regime of correlation times τ with the property $\tau \ll \bar{T}(\tau)$. Within this separation of time scales the rate concept is valid [21], meaning that $\rho(x,y,t)$ approaches an exponential decay $e^{-\bar{k}t}\,\rho(x,y)$ on a time scale that is negligible in comparison with $\bar{T}(\tau)$. It then readily follows from (3.10) and (3.13) that the decay rate $\bar{k}$ and the quasi-invariant density $\rho(x,y)$ satisfy

$$\bar{k}^{-1} = \bar{T}(\tau) \tag{5.1}$$

$$\Gamma\rho(x,y) = -\bar{k}\rho(x,y). \tag{5.2}$$

For the determination of the rate $\bar{k}$ we will use an approach based on the theory of quasipotentials. The mathematical foundation of this theory was elaborated mainly by Freidlin and Wentzell; see [25] for a detailed account of their work. Further important contributions are due to Graham and Tél [27] (see also [30] for a recent review) and Maier and Stein [32–34]. The general theoretical framework has been partially rederived and applied to particular model systems by Ludwig [26], Jauslin [28], Kautz [29], and Dykman and co-workers [31] to name but a few. Finally, concepts of quasipotential theory also entered into the investigation of bistable systems driven by colored noise through their close relation to the path integral formalism [42]. Here, we only sketch those parts of the general theory needed for the further investigations in Secs. VI–VIII: Expecting the usual Boltzmann-like structure of the quasi-invariant density $\rho(x,y)$, one starts with a WKB ansatz $\rho(x,y) = Z(x,y)\,e^{-\phi(x,y)/D}$, where the quasipotential $\phi(x,y)$ is assumed to be D independent and the prefactor $Z(x,y)$ may depend on D at most algebraically. In other words, the quasipotential represents the exponentially leading weak-noise behavior of the quasi-invariant density

$$\phi(x,y) = -\lim_{D\to 0} D\,\ln\rho(x,y). \tag{5.3}$$

Introducing the WKB ansatz for $\rho(x,y)$ into the master equation (5.2) and collecting terms of order $1/D$ then yields the Hamilton-Jacobi equation for the quasipotential

$$H(\partial_x\phi(x,y),\, \partial_y\phi(x,y),\, x,\, y) = 0\ , \tag{5.4}$$

$$H(p,q,x,y) := p^2 + q^2/\tau - U_y'(x)\,p - y\,q/\tau\ , \tag{5.5}$$

where p and q denote the canonical momenta conjugate to x and y, respectively. Similarly, the remaining terms of the master equation yield a differential equation for the prefactor $Z(x,y)$.

Since in the Hamilton-Jacobi equation (5.4) the quasipotential plays the role of the action in classical mechanics, a formal solution at any point $(\hat{x},\, \hat{y}) \in \mathbb{R}^2$ is given by

$$\phi(\hat{x},\, \hat{y}) = \min_{x(t),\, y(t)} \int_{-\infty}^{\hat{t}} L[x(t),\, y(t)]\, dt\ , \tag{5.6}$$

where the Lagrangian

$$L[x,\, y] = [\dot{x} + U_y'(x)]^2/4 + \tau\,[\dot{y} + y/\tau]^2/4 \tag{5.7}$$

follows from the Hamiltonian (5.5) by Legendre transformation. The minimization in (5.6) is over all paths $x(t)$, $y(t)$ starting at time $t = -\infty$ at the point attractor $(0,0)$ of the deterministic dynamics (2.1), (2.3) with $D = 0$ and ending at the point $(\hat{x},\, \hat{y})$ at a time $t = \hat{t}$ which can still be chosen arbitrarily. This initial condition for the paths $x(t)$, $y(t)$ and the fact that one has not only to extremize but to minimize in (5.6) does not strictly follow from the analogy with classical mechanics but only from the rigorous foundation of quasipotential theory [25,27,30]. There, one actually starts with (5.6) as a definition of the quasipotential and then proves that it satisfies the Hamilton-Jacobi equation (5.4) and the property (5.3). Once the generalized potential is known, the exponentially leading weak-noise behavior of the rate $\bar{k}$ can be extracted from (5.2) by standard methods [21]. For the mean escape time (5.1) one finally obtains

$$\lim_{D\to 0} D\,\ln\bar{T}(\tau) = \min_{\delta G_\tau}\phi(x,y) - \phi(0,0) =: \Delta\phi(\tau)\ , \tag{5.8}$$

where the minimization is performed on the basin boundary δG_τ of the point attractor $(0,0)$ introduced in Sec. II.

Equation (5.8) may be rewritten under the form

$$\bar{T}(\tau) = \zeta(\tau, D)\, e^{\Delta\phi(\tau)/D} \tag{5.9}$$

with a pre-exponential factor $\zeta(\tau, D)$ that depends more weakly than the exponentially leading Arrhenius factor $e^{\Delta\phi(\tau)/D}$ on D in the weak-noise limit $D \to 0$. In fact, the findings of Refs. [33,34,44] suggest that $\zeta(\tau, D)$ converges towards a finite value for $D \to 0$ and any fixed τ. Under the additional tacit assumption that the τ dependence of this limiting function $\zeta(\tau, 0)$ is sufficiently smooth we thus expect that *the qualitative features (monotonicity, extrema) of $\Delta\phi(\tau)$ as a function of τ carry over to $\bar{T}(\tau)$ for sufficiently small D*. In the same way as the exponentially leading part $\Delta\phi(\tau)$ in (5.9) follows from the quasipotential (5.8), the pre-exponential factor $\zeta(\tau, D)$ can be readily determined, once the differential equation mentioned below (5.5) for the prefactor $Z(x, y)$ of the quasi-invariant density $\rho(x, y)$ is solved. However, it is only very recently that the full complexity of this problem has been clearly recognized [34] and we will restrict ourselves to the investigation of $\Delta\phi(\tau)$ in the sequel.

It is rather obvious from (5.6) and will be confirmed in the following sections that $\Delta\phi(\tau)$ in (5.8) is always a positive quantity. It follows that the separation of time scales $\tau \ll \bar{T}(\tau)$ is guaranteed for any τ in the weak-noise limit $D \to 0$ and thus the rate and quasipotential concepts are valid. It is only for finite (but small) D that they may break down if τ becomes very large.

According to Sec. IV B the kinetic model predicts $\bar{T}(\tau) \geq \bar{T}_0$ and according to Sec. IV D the kinetic model is expected to yield accurate approximations for $\bar{T}(\tau)$ provided the noise strength D is small and $\tau \gg \ln 1/D$. With Eq. (4.12) it follows that for sufficiently small D there exists a rather extended regime of correlation times τ, namely $\ln 1/D \ll \tau \ll \bar{T}(\tau)$, for which both the rate concept and the kinetic model are supposed to yield accurate and thus essentially identical approximations for $\bar{T}(\tau)$. In particular, *for small D the rate concept and the kinetic model provide a complete description of $\bar{T}(\tau)$*.

For weak noise D the mean escape time $\bar{T}(\tau)$ can be determined in the τ regime $\ln 1/D \ll \tau \ll \bar{T}(\tau)$, where both the rate and the kinetic descriptions are valid, as follows [20]: Since the kinetic model applies (see Sec. IV A) a particle experiencing an instantaneous quasistatic potential $U_y(x)$ will escape at a rate $k(y)$ given in (4.1). Since the characteristic time scale τ of the potential fluctuations y is much smaller than the typical escape time $\bar{T}(\tau)$ the distribution of the "quasistatic" potentials $U_y(x)$ is governed by the stationary distribution $\rho(y)$ of the Ornstein-Uhlenbeck process (2.3). The hence following mean escape rate $\bar{k} = \int_{-\infty}^{\infty} k(y)\, \rho(y)\, dy$ is equal to $[\bar{T}(\tau)]^{-1}$ thanks to the rate concept, see Eq. (5.1). With (4.7) we finally obtain

$$\bar{T}(\tau) = \bar{T}_0 \text{ for } \ln 1/D \ll \tau \ll \bar{T}_0\ . \tag{5.10}$$

This result together with (4.11) and (4.12) is complementary to (4.6) and (4.27) since it includes the range of its validity without additional conditions as in (4.26) but does not include the leading finite-D and finite-τ corrections. The prediction (5.10) compares quite well with the numerical simulations shown in Figs. 1–4 in view of the fact that D is not yet that small in these examples.

From (4.11), (5.9), and (5.10) the large-τ limit of $\Delta\phi(\tau)$ readily follows, while the pre-exponential factor $\zeta(\tau, D)$ indeed approaches a D- and τ-independent finite value, as expected below (5.9). Finally, we can infer from the above-mentioned fact that the rate and quasipotential concepts are valid for any finite τ in the weak-noise limit and the comparison of $\bar{T}_0$ and $\bar{T}(\infty)$ below (4.25) that *the limits $D \to 0$ and $\tau \to \infty$ do not commute* and *the rate concept fails for finite D in the limit $\tau \to \infty$* [13] unless the conditions (4.24), and (4.25) are satisfied.

It is plausible that if the rate concept fails for a certain $\tau = \tau_0$ then it stays invalid for all $\tau \geq \tau_0$. We thus expect that $\bar{T}(\tau)$ is bounded from above (in order of magnitude) by τ for $\tau \geq \bar{T}_0$, in agreement with Figs. 1–4 and our previous prediction that (4.6) is actually an upper bound.

VI. GENERAL PROPERTIES OF THE QUASIPOTENTIAL

From the definition of the quasipotential (5.6) it follows that $\phi(0,0) = 0$. Furthermore, one can infer from (5.7) that the Lyapunov property [25,27,30]

$$\frac{d}{dt}\phi(\tilde{x}(t), \tilde{y}(t)) \leq 0 \tag{6.1}$$

is satisfied along any solution $\tilde{x}(t)$, $\tilde{y}(t)$ of the deterministic dynamics (2.1), (2.3) with $D = 0$. This property and the fact that on the basin boundary δG_τ the saddle point $(1,0)$ is the unique attractor of the deterministic dynamics (see Sec. II) imply that the minimum in (5.8) is taken for $x = 1$, $y = 0$ and thus $\Delta\phi(\tau) = \phi(1,0)$. Since $(1,0)$ is a fixed point of the deterministic dynamics, a minimizing path $x(t)$, $y(t)$ in (5.6) with endpoint $x(\hat{t}) = \hat{x} = 1$, $y(\hat{t}) = \hat{y} = 0$ can stay there for an arbitrary period of time $t > \bar{t}$ without changing the value of $\phi(1,0)$. We thus can set $\hat{t} = \infty$ without loss of generality, i.e., $\Delta\phi(\tau)$ can be determined from

$$\Delta\phi(\tau) = \min_{x(t), y(t)} \int_{-\infty}^{\infty} L[x(t), y(t)]\, dt \tag{6.2}$$

with boundary conditions $x(\infty) = 1$ and $x(-\infty) = y(\pm\infty) = 0$.

The minimizing $x(t)$, $y(t)$ in (6.2) is called the *most probable escape path* (MPEP) [32,33]. Note that with $x(t)$, $y(t)$ also $x(t + \Delta t)$, $y(t + \Delta t)$ minimizes (6.2) for any real Δt. The uniqueness [34] of the MPEP can be restored for instance by introducing the additional condition that

$$x(t = 0) = 1/2. \tag{6.3}$$

In the weak-noise limit $D \to 0$ every successful escape attempt follows this path after an appropriate shift of the time scale [25–27,30]. For small but finite D the MPEP still gives the correct qualitative behavior of a typical successful escape attempt except for very large positive and negative times t, i.e., close to the point attractor $(0,0)$ and the saddle $(1,0)$ [33]. In particular, the successful escape attempts give rise to a distribution of exit points on the basin boundary δG_τ that is concentrated in a small neighborhood of the saddle $(1,0)$ [33,34] but generically is neither maximal nor symmetric about $(1,0)$ [32] (saddle point avoidance). Moreover, close to $(0,0)$ and $(1,0)$ a typical successful escape attempt resembles a free random walk rather than the smooth MPEP $x(t)$, $y(t)$ and, as a consequence, is of finite duration, in contrast to the infinite time that the MPEP $x(t)$, $y(t)$ needs to go from $(0,0)$ to $(1,0)$ [33]. Once the successful escape attempt has crossed the small neighborhood of the saddle $(1,0)$ it essentially follows a deterministic path along the x axis and finally is absorbed at the threshold $x_{\text{th}} > 1$.

For the sake of convenience, we will solve the variational problem (6.2) not by means of the corresponding Euler-Lagrange equations but the equivalent Hamilton equations

$$\dot{x} = \partial_p H = 2\,p - U'(x) - y\,W'(x), \tag{6.4}$$

$$\dot{p} = -\partial_x H = [U''(x) + y\,W''(x)]\,p, \tag{6.5}$$

$$\dot{y} = \partial_q H = 2\,q/\tau - y/\tau, \tag{6.6}$$

$$\dot{q} = -\partial_y H = W'(x)\,p + q/\tau\ , \tag{6.7}$$

where arguments t have been omitted and the Hamiltonian H is given by (5.5). The boundary conditions are $x(t=-\infty) = 0$ and $x(t=\infty) = 1$, while $p(t)$, $y(t)$, and $q(t)$ must vanish for $t = \pm\infty$. From the analogy with classical mechanics it is clear that one integration can be saved by taking into account that the Hamiltonian is a constant of motion. By focusing on $t = -\infty$ one sees that this constant is zero, i.e.,

$$H(p(t),\, q(t)\, x(t),\, y(t)) = 0 \text{ for all } t\ . \tag{6.8}$$

Moreover, it will often be useful to consider the auxiliary variable

$$r(t) := q(t) - y(t) \tag{6.9}$$

instead of $y(t)$, satisfying

$$\dot{r} = W'(x)\,p - r/\tau \tag{6.10}$$

with boundary conditions $r(\pm\infty) = 0$. The unique (formal) solutions of (6.7) and (6.10) fulfilling the correct boundary conditions are

$$q(t) = -\int_t^{\infty} e^{(t-s)/\tau}\, W'(x(s))\, p(s)\, ds\ , \tag{6.11}$$

$$r(t) = \int_{-\infty}^{t} e^{(s-t)/\tau}\, W'(x(s))\, p(s)\, ds\ . \tag{6.12}$$

Next we introduce *potential* $W(x)$ *of type I* [20] defined by the property that $W'(x)$ does not change sign and is not identically zero on the interval $[0,1]$. Without loss of generality we will always assume that $W(x)$ is monotonically increasing for $0 \le x \le 1$ and, in particular, that $\Delta W = W(1) - W(0) > 0$. A typical example is (2.6). We now want to prove that $\Delta\phi(\tau)$ is a decreasing function of τ for this type of potential $W(x)$. To this end we focus on an arbitrary but fixed τ value and denote by $x(t)$, $y(t)$ the corresponding MPEP, i.e., the minimizing path in (6.2). Since, in general, this path no longer minimizes (6.2) if we change τ, we obtain $\Delta\phi(\tau+\delta\tau) \le \int_{-\infty}^{\infty} L[x(t), y(t)]dt$ for any $\delta\tau$. By differentiation of $L[x,y]$ in (5.7) with respect to τ it then follows that

$$\Delta\phi(\tau+\delta\tau) \le \Delta\phi(\tau) + \frac{\delta\tau}{4}\int_{-\infty}^{\infty}\{[\dot{y}(t)]^2 - [y(t)/\tau]^2\}\,dt + O(\delta\tau^2)\ . \tag{6.13}$$

From Hamilton's equation (6.5) it is clear that $p(t)$ cannot change sign. In Appendix A it is shown that actually $p(t) > 0$ for any finite t and that $x(t)$ never leaves the domain with $W'(x) \ge 0$. With (6.11), (6.12) this implies that $q(t) - r(t) < 0$ for any finite t and $q(t)\,r(t)$ is nonpositive and not identically zero (provided $\tau > 0$). Moreover, from (6.6) and (6.9) one can infer that $\dot{y}^2 - (y/\tau)^2 = 4\,q\,r/\tau^2$. Therefore, it follows from (6.13) that $d\Delta\phi(\tau)/d\tau < 0$, i.e., $\Delta\phi(\tau)$ *is strictly monotonically decreasing with* τ *for any potential* $W(x)$ *of type I*, in agreement with the numerical example in Fig. 1. Finally, since $q(t) - r(t) < 0$ we find with (6.9) that the MPEP $x(t)$, $y(t)$ is contained in the half plain $y < 0$ for any finite t. In other words, *for potentials* $W(x)$ *of type I a particle (2.1) typically escapes while the potential* $U_y(x)$ *is in a "low state"* $y < 0$, see Fig. 5. More precisely, this property $y < 0$ only concerns the "intermediate phase" of a successful escape attempt while the particle x crosses the domain between 0 and 1: As mentioned below Eq. (6.3), for x very close to 0 and 1 and for $x \ge 1$, the typical y values become very small, possibly even with fluctuations into the positive domain. Similar features of typical successful escape attempts in the case of dichotomous potential fluctuations $y(t)$ in (2.1) have been found numerically in Ref. [2].

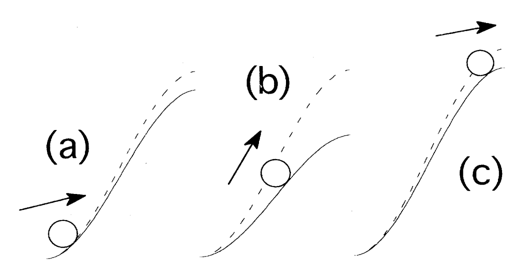

FIG. 5. Sketch of a typical successful escape attempt for a potential $W(x)$ of type I and $0 < \tau \ll \bar{T}(\tau)$ at three successive time instances (a)–(c). Solid lines: fluctuating potential $U_y(x) = U(x) + y\,W(x)$ for $0 \le x \le 1$; dashed lines: unperturbed potential $U(x)$; arrows: motion of the escaping particle.

VII. QUASIPOTENTIALS FOR SMALL $W(x)$

In this section we study the exponentially leading weak-noise behavior of the mean escape time (5.8) and the MPEP for asymptotically small potentials $W(x)$. It is convenient to introduce an explicit small parameter γ by means of the substitution

$$W(x) \mapsto \gamma\, W(x). \tag{7.1}$$

This suggest an expansion of the minimizing $x(t)$ in (6.2) as $x_0(t) + \gamma\, x_1(t) + \cdots$ and similarly for $y(t)$ and their canonically conjugated momenta $p(t)$ and $q(t)$. The equations governing $x_0(t)$, $x_1(t)$,... can then be found by comparing powers of γ in Hamilton's equations (6.4)–(6.7) [or their formal solutions (6.11), (6.12)] and in the energy conservation law (6.8). The boundary conditions become $x_0(t=\infty)=1$ and $x_0(t=-\infty)=x_i(t=\pm\infty)=0$ for $i\geq 1$, while $y_i(t)$, $p_i(t)$, and $q_i(t)$ must vanish for $t=\pm\infty$ and any $i\geq 0$. The additional condition (6.3) requires that $x_0(t=0)=1/2$ and $x_i(t=0)=0$ for $i\geq 1$.

In order γ^0 one sees from (6.9), (6.11), (6.12) that $y_0(t)$ and $q_0(t)$ identically vanish. Then Hamilton's equation (6.4) and the energy conservation law (6.8) lead to $\dot{x}_0 = 2\,p - U'(x_0)$ and $p_0\,[U'(x_0)-p_0]=0$, where we omitted arguments t. In view of the boundary conditions $x_0(t=-\infty)=0$ and $x_0(t=\infty)=1$ this implies

$$\dot{x}_0 = p_0 = U'(x_0). \tag{7.2}$$

Since we assumed above Eq. (2.5) that $U'(x)$ has no zero for $0<x<1$ it follows that $x_0(t)$ is a strictly monotonically increasing function of t with $x_0(t=-\infty)=0$, $x_0(t=0)=1/2$, and $x_0(t=\infty)=1$.

In order γ^1 one readily finds $q_1(t)$ and $r_1(t) := q_1(t) - y_1(t)$ by introducing $x_0(t)$ and $q_0(t)$ into (6.11), (6.12). Since $x_0(t)$ increases strictly monotonically, we can use $x_0\in[0,1]$ instead of $t\in\mathbb{R}$ as parameter. For $q_1(x_0) := q_1(t(x_0))$ and $r_1(x_0) := r_1(t(x_0))$ one then obtains

$$q_1(x_0) \;=\; -\int_{x_0}^{1} W'(y)\,K(y,x_0)\,dy\ , \tag{7.3}$$

$$r_1(x_0) \;=\; \int_0^{x_0} W'(y)\,K(x_0,y)\,dy\ , \tag{7.4}$$

$$K(x,y) := \exp\left\{-\frac{1}{\tau}\int_y^x \frac{dz}{U'(z)}\right\}\ . \tag{7.5}$$

In particular, the correct boundary conditions are satisfied, i.e., $q_1(x_0)$ and $y_1(x_0)=q_1(x_0)-r_1(x_0)$ vanish for $x_0=0$ and $x_0=1$. Introducing our results so far into Hamilton's equation (6.4) and the energy conservation law (6.8) yields $x_1'(x_0) = x_1(x_0)\,U''(x_0)/U'(x_0)$ and $p_1(x_0) = U''(x_0)\,x_1(x_0)$. The general solution of the first equation is $x_1(x_0) = \text{const}\times U'(x_0)$. Due to the condition $x_1(t=0)=0$, or equivalently, $x_1(x_0=1/2)=0$ one finally finds that $x_1(x_0)$ and $p_1(x_0)$ identically vanish.

In summary, we have found a unique solution of the variational problem (6.2) which can be expanded in powers of γ, at least up to the order γ^1. By similar methods a straightforward calculation yields a unique solution up to the order γ^2. In other words, there exists exactly one solution of Hamilton's equations (6.4)–(6.7) plus boundary conditions of the form $x(t)=x_0(t)+\gamma\,x_1(t)+O(\gamma^2)$ and similarly for $y(t)$, $p(t)$, and $q(t)$. However, in principle, there might be further solutions that are not of this form. In this case, we tacitly assume that they correspond to relative extrema but not to the absolute minimum in (6.2). Although we cannot prove this assumption, the agreement of our later result for $\Delta\phi(\tau)$ with known limiting cases strongly suggests its validity.

As detailed in Appendix B, by introducing our above results for the MPEP $x(t)$, $y(t)$ into (6.2) one obtains for the exponentially leading weak-noise behavior of the mean escape time (5.8)

$$\begin{aligned}\Delta\phi(\tau) &= \Delta U - \gamma^2 I(\tau) + O(\gamma^4),\\ I(\tau) &= \int_0^1 dx \int_x^1 dy\, W'(x)\,W'(y)\\ &\quad\times \exp\left\{-\frac{1}{\tau}\int_x^y \frac{dz}{U'(z)}\right\}\ .\end{aligned} \tag{7.6}$$

For small correlation times one readily sees that $I(\tau)$ becomes $\tau\int_0^1 [W'(x)]^2 U'(x)\,dx$, in agreement with (3.7). Furthermore, by differentiation of $\Delta\phi(\tau)$ in (7.6) with respect to τ one recovers for small γ the general result from Sec. VI that $\Delta\phi(\tau)$ is strictly monotonically decreasing with τ for any potential $W(x)$ of type I.

In Figs. 4(a) and 4(b) the approximate prediction

$$\bar{T}(\tau) \simeq \bar{T}(0)\, e^{-\gamma^2 I(\tau)/D} \tag{7.7}$$

following from (7.6) and (5.9) is compared with numerical simulations. While the qualitative agreement is satisfactory, the quantitative differences can be attributed either to the unknown corrections of order $O(\gamma^4)$ in (7.6) or the unknown τ dependence of the pre-exponential factor $\zeta(\tau,D)$ in (5.9) which both have been neglected in (7.7). From the obvious invariance of $\bar{T}(\tau)$ under $\gamma\mapsto-\gamma$ we can conclude that $\zeta(\tau,D)/\zeta(0,D)$ must be of the form $1-\gamma^2\zeta_1(\tau,D)+O(\gamma^4)$. Neglecting terms of order $O(\gamma^4)$ we thus obtain from (7.6) and (5.9) that the unknown function $\zeta_1(\tau,D)$ should satisfy

$$\zeta_1(\tau,D) = \frac{1}{\gamma^2}\left(1-\frac{\bar{T}(0)}{\bar{T}(\tau)}\,e^{-\gamma^2 I(\tau)/D}\right) \tag{7.8}$$

independent of γ. This prediction is verified in Fig. 4(c), showing that corrections of order $O(\gamma^4)$ are indeed negligible for the examples shown in Fig. 4. In order to show that the deviations in Figs. 4(a) and 4(b) from (7.7) are a mere prefactor effect one has to consider smaller noise-strengths D. Since this is numerically very expensive, we restricted ourselves to a single τ value close to the minimum of $\bar{T}(\tau)$, see Fig. 4(d). According to (5.8) the quantity $\ln\bar{T}(\tau)$ as a function of $1/D$ approaches a straight line for large values of $1/D$ with slope $\Delta\phi(\tau)$ [45]. The simulations shown in Fig. 4(d) strongly support the validity of the theoretical prediction (7.6) for this slope $\Delta\phi(\tau)$. However, for a completely convincing verification of (7.6) one should either consider still smaller D and γ values or find an analytic approximation for $\zeta(\tau,D)$. Unfortunately, the former requires a very large numerical effort, while the latter is a very difficult problem as well [34], as already mentioned below (5.9), being resistant to all our analytical attempts so far.

For large τ it is shown in Appendix C that

$$I(\tau) = \frac{\Delta W^2}{2} - \frac{1}{\tau}\int_0^1 \frac{[W(1)-W(x)]\,[W(x)-W(0)]}{U'(x)}\,dx + O(\tau^{-2}) \ . \tag{7.9}$$

As predicted by (5.9), (5.10) the exponentially leading part of $\bar{T}_0$ from (4.16) is recovered from Eqs. (7.6), (7.9) for asymptotically large τ [note that κ in (4.16) is of the order $O(\gamma^4)$]. Equation (7.9) motivates the introduction of *potentials* $W(x)$ *of type II* defined by the property that $\Delta W = 0$. A typical example is (2.7). Since only the derivative $W'(x)$ plays a role in Eq. (2.1) we will henceforth always assume that $W(0) = W(1) = 0$ without loss of generality. For potentials $W(x)$ of type II we then find from (7.6) and (7.9) in leading order γ and $1/\tau$ that

$$\Delta\phi(\tau) = \Delta U - \frac{\gamma^2}{\tau}\int_0^1 \frac{[W(x)]^2}{U'(x)}\,dx. \tag{7.10}$$

In contrast to the type I case, $\Delta\phi(\tau)$ *is thus monotonically increasing for large* τ. Since $\Delta\phi(\tau)$ is always decreasing for small τ [see Eq. (3.7) and the discussion below Eq. (7.6)] and $\Delta\phi(\infty) = \Delta\phi(0)$ we can infer that $\Delta\phi(\tau)$ *has an absolute minimum at a finite correlation time* $\tau_{\rm RA}$ (the index RA refers to "resonant activation"). Although $\tau_{\rm RA}$ can, in general, not be determined explicitly by means of (7.6), a comparison of the small- and large-τ asymptotics (3.7) and (7.10) suggests that $\tau_{\rm RA}$ is comparable to the "typical inverse slope" of $U(x)$ in the region $0 \le x \le 1$,

$$\tau_{\rm RA} = O(1/\Delta U) \ \ \text{(type II)} \ . \tag{7.11}$$

By closer inspection one finds that in a small-τ expansion of (7.6) the terms of order τ and τ^2 are of opposite sign and become of comparable magnitude for τ values of the order $1/\Delta U$ and similarly for a large-τ expansion. This is a further confirmation of (7.11). Very simple intuitive arguments leading to the same conclusion (7.11) have been given in Ref. [20]. In Sec. VIII we will see that these properties of type II potentials, in particular (7.10) and (7.11), stay valid even if the fluctuating part of the potential $\gamma\, W(x)$ is no longer small; see also Figs. 2 and 4.

Potentials $W(x)$ that are neither of type I nor II are called of *mixed type*. They have the property that $\Delta W \neq 0$ and $W(x)$ has at least one extremum in the region $0 < x < 1$. As a particular toy model we consider a piecewise linear continuous $W(x)$ respecting

$$W'(x) = \begin{cases} 0 & \text{for } x < 1/5 \text{ and } x > 4/5 \\ w_1 & \text{for } 1/5 < x < 2/5 \\ w_2 & \text{for } 2/5 < x < 3/5 \\ w_3 & \text{for } 3/5 < x < 4/5, \end{cases} \tag{7.12}$$

where w_1, w_2, w_3 are parameters, see Fig. 6(a). The nondifferentiabilities are of no relevance since they can be made smooth without notably changing the properties of $\Delta\phi(\tau)$. For a potential $U(x)$ with a constant slope in the domain $1/5 < x < 4/5$ the double integral in (7.6) can be performed explicitly. Depending on the parameter values w_1, w_2, w_3 in (7.12), different qualitative features of $I(\tau)$ and thus of $\Delta\phi(\tau)$ are found, see Fig. 6(b). First, $I(\tau)$ is monotonically increasing if $W(x)$ is of type I but additionally for some $W(x)$ of mixed type. Second, $I(\tau)$ shows a maximum at $\tau_{\rm RA} = O(1/\Delta U)$ if $W(x)$ is of type II but also for some $W(x)$ of mixed type. In the latter case $I(\tau)$ converges *from above* towards the positive limit $\Delta W^2/2$ for $\tau \to \infty$. Finally, there are potentials $W(x)$ of mixed type for which $I(\tau)$ exhibits a maximum in the domain $\tau = (1/\Delta U)$ followed by a relative minimum and then monotonically *increases* towards $\Delta W^2/2$ for large τ. We mention that the latter type of $I(\tau)$ would not be observed if $W(x)$ consisted only of four linear pieces. The consequences of these findings will be discussed in more detail in Sec. IX. Obviously, the example shown in Fig. 3 was inspired by the properties of $I(\tau)$ for the piecewise linear toy model (7.12). This demonstrates the

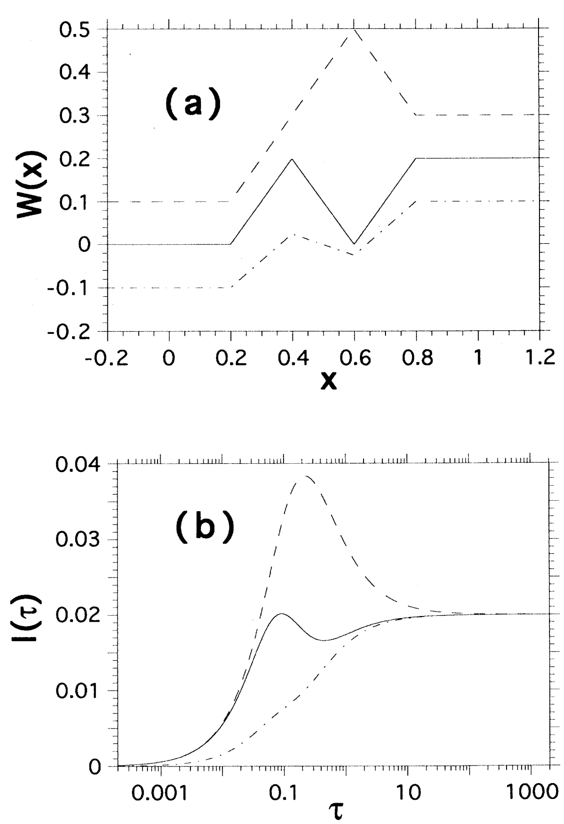

FIG. 6. (a) Three examples of potentials $W(x)$ defined by (7.12) with parameters $w_1 = w_3 = 1$ and $w_2 = -1$ (solid line), $w_1 = w_2 = 1$ and $w_3 = -1$ (dashed line), $w_1 = w_3 = 0.625$ and $w_2 = -0.25$ (dashed-dotted line). Note that $W(0)$ can be chosen arbitrarily since only $W'(x)$ enters the dynamics (2.1); (b) The corresponding functions $I(\tau)$ from (7.6) for a potential $U(x)$ with a constant slope $U'(x) = 1$ in the domain $1/5 \le x \le 4/5$.

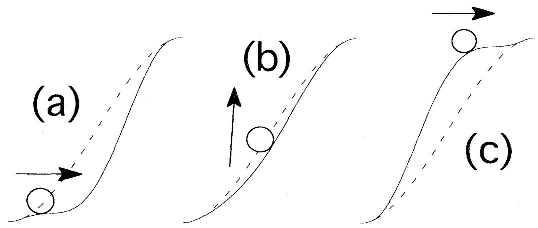

FIG. 7. Same as Fig. 5 but for a potential $W(x)$ of type II with a single hump on the interval $[0,1]$ [49].

predictive power of (7.6) in spite of our ignorance of the pre-exponential factor $\zeta(\tau, D)$ in (5.9).

We finally discuss the MPEP $x(t)$, $y(t)$ as introduced in the second paragraph of Sec. VI. With x_0 instead of t as parameter and omitting terms of order γ^2 the MPEP is given by $x(x_0) = x_0$ and

$$y(x_0) = \gamma\, y_1(x_0) = \gamma\, [q_1(x_0) - r_1(x_0)] \ . \tag{7.13}$$

We are thus left with the discussion of Eqs. (7.3)–(7.5). For potentials of type I one immediately recovers $y(x_0) < 0$ for $0 < x_0 < 1$; see the end of Sec. VI and Fig. 5. Next we address potentials of type II *with a single hump*, i.e., with a single zero $\bar{x}$ of $W'(x)$ in the domain $0 < x < 1$. Without loss of generality we further assume that $W(x)$ is non-negative for $0 \le x \le 1$, in particular $W(\bar{x}) > 0$. For such a potential $W(x)$ it is shown in Appendix D that $y_1(x_0)$ has exactly one zero (besides the trivial ones at $x_0 = 0$ and $x_0 = 1$) and is negative below and positive beyond this zero. In other words, we recover the suggestive escape mechanism [20] as sketched in Fig. 7: Typically, a successful escape attempt leaves the neighborhood of $x = 0$ while the potential $U_y(x)$ is in a "low" state, $y < 0$ [Fig. 7(a)], then the particle is lifted by a large fluctuation of y [Fig. 7(b)], and finally it moves in a potential $U_y(x)$ in a "high" state $y > 0$ towards the potential barrier $x = 1$ [Fig. 7(c)].

For potentials $W(x)$ of type II with more than a single hump on $[0,1]$ and of mixed type the behavior of $y_1(x_0)$ becomes more involved and is not discussed in further detail here. We only mention that for small τ one finds from (7.3)–(7.5) that

$$y_1(x_0) = -2\,\tau\, W'(x_0)\, U'(x_0) + O(\tau^2) \ . \tag{7.14}$$

Similarly as for $\Delta\phi(\tau)$, it turns out that this small-τ result stays valid even if the fluctuating part of the potential $\gamma\, W(x)$ is no longer small. In other words, the corrections of order $O(\gamma^4)$ in (7.6) and of order $O(\gamma^2)$ in (7.13) vanish for $\tau \to 0$.

VIII. QUASIPOTENTIALS FOR LARGE τ

In this section we investigate $\Delta\phi(\tau)$ from (6.2) for large correlation times τ. Unlike in the preceding section, here the fluctuating part of the potential is not required to be small and thus we do *not* make the substitution (7.1). In view of (5.9), (5.10) we expect that $\Delta\phi(\tau)$ approaches the exponentially leading part of $\bar{T}_0$ from (4.11) in the limit $\tau \to \infty$. According to the discussion following Eq. (4.11), an explicit evaluation of $\Delta\phi(\tau)$ even in the limit $\tau \to \infty$ will thus not be possible without any further assumption regarding $W(x)$. In order to keep things simple at least in the limit $\tau \to \infty$ we, therefore, restrict ourselves to potentials $W(x)$ such that $V(x)$ defined in (4.15) is strictly monotonically increasing for $0 < x < 1$ and $W'(0) = W'(1) = 0$. In view of (4.16) we thus expect that $\Delta\phi(\tau)$ converges towards $\Delta U - \Delta W^2/2$ for $\tau \to \infty$.

Next we introduce the following three assumptions regarding the MPEP $x(t)$, $y(t)$ and the canonically conjugate momenta $p(t)$, $q(t)$: (i) $\dot{x}(t)$ is positive and thus $x(t)$ is strictly monotonically increasing for any finite t. (ii) the difference

$$\delta p(t) := \dot{x}(t) - p(t) \tag{8.1}$$

tends to zero for $\tau \to \infty$. (iii) $y(t)$ and $q(t)$ are bounded in the limit $\tau \to \infty$, i.e., there exists a constant C with $|y(t)| \le C$ for any t and any sufficiently large τ, and similarly for $q(t)$. In the sequel we will determine a unique solution of Hamilton's equations (6.4)–(6.7) plus boundary conditions that are consistent with these assumptions (i)–(iii). However, in principle, we cannot exclude that there exist still other solutions that do not satisfy (i)–(iii). If so, we tacitly assume that our solution is the absolute minimum in (6.2). As a justification we anticipate that our final result (8.11) agrees with all known limiting cases, namely (4.16) for $\tau \to \infty$ and (7.6), (7.9) for asymptotically small $W(x)$. We also find the assumption (i) plausible since we would not expect that a particle $x(t)$ stops or changes its direction during a typical successful escape attempt. We further note that the assumption (iii) is equivalent to the boundedness of any two of the functions $y(t)$, $\tau\,\dot{y}(t)$, $q(t)$, and $r(t)$ in the limit $\tau \to \infty$ due to (6.6), (6.7), (6.9). The boundedness of $y(t)$ is plausible since in (2.4) an infinite y has zero probability independent of τ. The boundedness of $y(t)$ and $q(t)$ is also strongly suggested by Hamilton's equations (6.4)–(6.7) though we are not able to give a rigorous proof. The assumption (ii) that $\dot{x}(t) \to p(t)$ for $\tau \to \infty$ is, however, not so obvious.

The assumption (i) that $\dot{x}(t) > 0$ for $-\infty < t < \infty$ implies that there exists a function $\bar{V}(x)$ satisfying

$$\dot{x}(t) = \bar{V}'(x(t)). \tag{8.2}$$

Since $\dot{x}(t) > 0$ we may equally well use $x \in [0,1]$ instead of $t \in \mathbb{R}$ as parameter. Introducing (8.1), (8.2) into Hamilton's equation (6.4) then yields the exact identity

$$\bar{V}'(x) = U'(x) + y(x)\, W'(x) + 2\, \delta p(x) \ . \tag{8.3}$$

Since $\delta p(x)$ becomes small for large τ according to our assumption (ii), $\bar{V}'(x)$ can often be approximated by $U'(x) + y(x)\, W'(x)$ in the sequel. Even more, in our final result we will be able to replace $\bar{V}(x)$ by $V(x)$ from (4.15), which motivates the notation $\bar{V}(x)$ for the unknown function satisfying (8.2).

With (8.1), (8.2), and x as parameter we can rewrite Eqs. (6.11) and (6.12) under the form

$$q(x) = -\int_x^1 W'(y)\,[1 - \delta p(y)/\bar{V}'(y)]\,\bar{K}(y,x)\,dy, \quad (8.4)$$

$$r(x) = \int_0^x W'(y)\,[1 - \delta p(y)/\bar{V}'(y)]\,\bar{K}(x,y)\,dy\ , \quad (8.5)$$

$$\bar{K}(x,y) := \exp\left\{-\frac{1}{\tau}\int_y^x \frac{dz}{\bar{V}'(z)}\right\}\ . \quad (8.6)$$

Exploiting the assumptions (ii) and (iii) it is shown in Appendix E that

$$\delta p(x) = \frac{[W(1) - W(x)]\,[W(x) - W(0)]}{\tau\,\bar{V}'(x)} + O(\tau^{-2})\ . \quad (8.7)$$

Furthermore, it is shown in Appendix E that for most x values in (8.4), (8.5) the leading order behavior for large τ can be obtained by expanding the exponential function (8.6) in powers of $1/\tau$. More precisely, it is shown that

$$q(x) = W(x) - W(1) + \int_x^1 \left[\frac{W(1) - W(y)}{\tau\,\bar{V}'(y)} + \frac{W'(y)\,\delta p(y)}{\bar{V}'(y)}\right] dy + o\left(\frac{1}{\tau^{2-\epsilon}}\right) \quad (8.8)$$

for $1 \geq x \geq x_\tau$, while in the domain $0 \leq x \leq x_\tau$ the function $q(x)$ rapidly grows from $q(0) = 0$ towards the value $q(x_\tau)$ from (8.8). Here, $o(1/\tau^{2-\epsilon})$ denotes an asymptotical decrease for large τ that is faster than $1/\tau^{2-\epsilon}$ for any $\epsilon > 0$ but slower than $1/\tau^2$ and the quantity x_τ is defined as $\exp\{-\tau^{\epsilon/3}\}$, tending to zero faster than any power of $1/\tau$ for $\tau \to \infty$. Similarly, one finds that

$$r(x) = W(x) - W(0) - \int_0^x \left[\frac{W(y) - W(0)}{\tau\,\bar{V}'(y)} + \frac{W'(y)\,\delta p(y)}{\bar{V}'(y)}\right] dy + o\left(\frac{1}{\tau^{2-\epsilon}}\right) \quad (8.9)$$

for $0 \leq x \leq 1 - x_\tau$ and that $r(x)$ decreases rapidly from $r(1 - x_\tau)$ given by (8.9) towards $r(1) = 0$ in the domain $1 \geq x \geq 1 - x_\tau$.

From (8.1) it is clear that $\delta p(x) \to 0$ for $x \to 0$. Even more, it is shown in Appendix E that there exists a constant $A_1 > 0$ with $\bar{V}'(x) > A_1\,x$ for sufficiently small x. Since $W'(0) = 0$, the difference $W(x) - W(0)$ in (8.7) decreases at least quadratically for small x and thus the convergence $\delta p(x) \to 0$ for $x \to 0$ goes proportional to x or even faster. Analogous properties hold for $x \to 1$. Using these properties of $\delta p(x)$ and the assumption (ii) that $\delta p(x) \to 0$ for $\tau \to 0$ it is not difficult to see that the term $2\,\delta p(x)$ in $\bar{V}'(x)$ from (8.3) can be consistently neglected as far as Eqs. (8.7)–(8.9) are concerned without changing the accuracy of order $O(\tau^{-2})$ and $o(1/\tau^{2-\epsilon})$, respectively. For $y(x) = q(x) - r(x)$ [see Eq. (6.9)] it then follows from (8.7)–(8.9) that

$$y(x) = -\Delta W + O(1/\tau) \quad (8.10)$$

for $x_\tau \leq x \leq 1 - x_\tau$, while in the domain $0 \leq x \leq x_\tau$ the function $y(x)$ rapidly grows from $y(0) = 0$ towards $y(x_\tau)$ from (8.10), and similarly for $1 - x_\tau \leq x \leq 1$. So, *except for potentials $W(x)$ of type II an expansion of $y(x)$ in powers of $1/\tau$ about $\tau = \infty$ is impossible* since $y(x)$ becomes discontinuous at $x = 0$ and $x = 1$ for $\tau \to \infty$. Essentially, this is the reason for the rather involved calculations in the present section. It can be shown that this feature of $y(x)$ also agrees with the kinetic model from Sec. IV in the regime $\tau_{\min}(D) \leq \tau \ll \bar{T}(\tau)$ as predicted at the end of Sec. V: In this regime practically all particles (2.1) escape while they experience a "quasistatic" potential $U_y(x)$ with $y = -\Delta W$.

It follows from (8.10) that in (8.3) not only the term $2\,\delta p(x)$ can be neglected but also $y(x)$ can be approximated by $-\Delta W$ in (8.7)–(8.9) without changing the accuracy of order $O(\tau^{-2})$ and $o(1/\tau^{2-\epsilon})$, respectively. In other words, we can substitute $\bar{V}(x)$ by $V(x)$ from (4.15). In particular, one can readily verify now by means of (8.2), (8.3), and (8.7)–(8.9) the consistency of our results with the assumptions (i)–(iii) at the beginning of this section. It also follows that the assumptions that $W'(0) = W'(1) = 0$ and that $V(x)$ is strictly monotonically increasing for $0 < x < 1$ *cannot* be easily relaxed.

Using our results so far, the evaluation of $\Delta\phi(\tau)$ from (6.2) is straightforward. As detailed in Appendix F, one finally obtains

$$\Delta\phi(\tau) = \Delta U - \frac{\Delta W^2}{2} + \frac{E}{\tau} + o\left(\frac{1}{\tau^{2-\epsilon}}\right)$$

$$E := \int_0^1 \frac{[W(1) - W(x)]\,[W(x) - W(0)]}{U'(x) - \Delta W\,W'(x)}\,dx\ . \quad (8.11)$$

One recovers $\Delta\phi(\tau) = \Delta U - \Delta W^2/2$ for $\tau \to \infty$ as expected at the beginning of this section as well as (7.6), (7.9) for asymptotically small $W(x)$. The general prediction from Sec. VI that $\Delta\phi(\tau)$ is strictly monotonically decreasing for potentials $W(x)$ of type I is also satisfied. For potentials $W(x)$ of type II $\Delta W = 0$ and (8.11) exactly agrees with (7.10). As a consequence, the conclusions below Eq. (7.10) immediately generalize to arbitrary potentials $W(x)$ of type II with $W'(0) = W'(1) = 0$.

For a comparison of (8.11) with numerical simulations we neglect the contributions of order $o(1/\tau^{2-\epsilon})$ as well as the τ dependence of the unknown prefactor $\zeta(\tau, D)$ in (5.9), similarly as in (7.7). With (5.10) this yields the approximation

$$\bar{T}(\tau) = \bar{T}_0\,\exp\{E/(D\,\tau)\} \quad (8.12)$$

for large τ but with $\tau \ll \bar{T}_0$ which is in good agreement with the numerical results in Figs. 1, 2, and 4. [For the example in Fig. 3 the function $V(x)$ from (4.15) is not strictly monotonically increasing for $0 < x < 1$ and one indeed finds that (8.12) compares very badly with the numerical simulations.]

We finally discuss the MPEP, written in the form $y(x)$, for potentials $W(x)$ of type II. Since $W(0) = W(1) = 0$ one finds from (8.7)–(8.9) that

$$y(x) = \frac{1}{\tau}\left[2\int_0^x \frac{W(z)}{U'(z)}\,dz - Y\right] + o\left(\frac{1}{\tau^{2-\epsilon}}\right), \qquad (8.13)$$

$$Y = \int_0^1 \left[\frac{W(z)}{U'(z)} + \frac{W'(z)\,[W(z)]^2}{[U'(z)]^2}\right] dz \qquad (8.14)$$

for $x_\tau \leq x \leq 1 - x_\tau$. If both $U'(x)$ and $W(x)$ are symmetric about $x = 1/2$ for $0 \leq x \leq 1$ then (8.13), (8.14) simplify to

$$y(x) = \frac{2}{\tau}\int_{1/2}^x \frac{W(z)}{U'(z)}\,dz. \qquad (8.15)$$

For potentials $W(x)$ of type II with a single hump on [0, 1] we recover from (8.13)–(8.15) the typical escape mechanism discussed at the end of Sec. VII and sketched in Fig. 7, at least if either $W(x)$ is small or $U'(x)$ and $W(x)$ are symmetric about $x = 1/2$ for $0 \leq x \leq 1$. However, if neither of the latter conditions is satisfied one can find examples [even with a single humped $W(x)$] for which $y(x)$ from (8.13) does not change sign. In these examples the potential $W(x)$ must be sufficiently large such that the last term in (8.14) dominates (8.13). Hence, in general the typical escape mechanism may be more complicated even for single humped potentials $W(x)$ of type II than in Fig. 7.

IX. SUMMARY AND DISCUSSION

We studied the escape problem for an overdamped Brownian particle (2.1) in a fluctuating metastable potential in the presence of thermal noise. The potential fluctuations are driven by an Ornstein-Uhlenbeck process (2.3) in the stationary state with an invariant density (2.4) that is independent of the correlation time τ. Regarding the fluctuating part $W(x)$ of the potential we distinguished three different types: type I if $W'(x)$ does not change sign between the well $x = 0$ and the barrier $x = 1$ of the metastable static part $U(x)$ of the potential (without loss of generality one can assume that $W(x)$ is monotonically increasing on [0,1]), type II if $\Delta W = W(1) - W(0)$ is zero (without loss of generality one can assume that $W(0) = W(1) = 0$), and a mixed type if $W(x)$ is neither of type I nor II.

As our central quantity we introduced the mean escape time $\bar{T}$ across a threshold $x_{\rm th}$ beyond the barrier $x = 1$ of $U(x)$ for an ensemble of particles (2.1) with an initial distribution $\rho_0(x)$ concentrated about the well $x = 0$ of $U(x)$. Of particular interest is the dependence of the mean escape time $\bar{T}(\tau)$ on the correlation time τ of the potential fluctuations, especially the possible occurrence of "resonant activation" [minimum of $\bar{T}(\tau)$ in the domain $0 < \tau < \infty$]. For asymptotically small and infinite τ we derived the general properties (3.1)–(3.6) of $\bar{T}(\tau)$. As a consequence, we found as sufficient conditions for the existence of resonant activation that either $W'(x)$ must vanish whenever $U'(x) < 0$ and $x \leq x_{\rm th}$ or that the noise strength D must be small. These findings compare very well with the numerical simulations from Figs. 1–4.

In the following we restricted ourselves to the most interesting case of weak noise D. In this case the escape problem is governed by three relevant time scales: the characteristic time scale τ of the potential fluctuations that plays the role of a control parameter and may vary between 0 and ∞, the mean escape time $\bar{T}(\tau)$ that is governed by an exponentially leading Arrhenius factor for weak noise D, and the time scale T_a of the escape attempts or, equivalently, of the intrawell relaxation that increases like $\ln 1/D$ for weak noise D.

For small D and $\tau \gg T_a$ we introduced a "kinetic model" by means of an adiabatic elimination procedure in the full escape problem. The kinetic model itself is well defined for arbitrary τ and D. Within this model, $\bar{T}(\tau)$ is constantly equal to $\bar{T}(0)$ from (3.8) if D is asymptotically small and the conditions (4.24) and (4.25) are satisfied, i.e., for a subclass of type II potentials $W(x)$. In any other case, $\bar{T}(\tau)$ is strictly monotonically increasing with τ from $\bar{T}_0$ towards $\bar{T}(\infty)$ defined in (4.7) and (4.9), respectively, and $\bar{T}(\infty)$ is strictly larger than $\bar{T}(0)$ from (3.8). For small and large τ the leading order approximations for $\bar{T}(\tau)$ are given by (4.6) and (4.8) and we showed (within certain restrictions) that they should be valid for weak noise D at least as long as τ is much smaller than $\bar{T}(0)$ from (3.8) and much larger than $\bar{T}(\infty)\,e^{\Delta W^2/D}$, respectively. We finally showed that these results for $\bar{T}(\tau)$ from the kinetic model provide accurate approximations for the true mean escape time of a particle (2.1) (i.e., the above-mentioned adiabatic elimination is justified) under the necessary conditions that D is small and $\tau \geq \tau_{\rm min}(D)$ with $\tau_{\rm min}(D)$ diverging much faster than $T_a \sim \ln 1/D$ but much slower than $\bar{T}_0$ for $D \to 0$. We gave arguments suggesting that these conditions are, in fact, sufficient, in agreement with the simulations shown in Figs. 1–4. Consequently, the kinetic models put forward in Refs. [4,5] are *not* a valid description of the escape problem (2.1) considered here in the rather extended regime $\tau \leq \tau_{\rm min}(D)$. In particular, they are not suitable to predict the existence of resonant activation nor do they capture the essential escape mechanisms giving rise to this effect for the model considered in this paper, as already pointed out in Ref. [5].

For sufficiently small D and $\tau \ll \bar{T}(\tau)$ the rate concept can be used. In particular, in the weak-noise limit $D \to 0$ the rate concept is valid for any finite τ. Since $\bar{T}(\tau) \gg T_a$, for small (but finite) D the rate concept and the kinetic model provide a complete (approximate) description of $\bar{T}(\tau)$. In the rather extended regime $\ln 1/D \ll \tau \ll \bar{T}(\tau)$ where both the rate and the kinetic descriptions are valid, the mean escape time is approximately constant; see (5.10). However, in general, the rate concept breaks down if $\tau \ll \bar{T}(\tau)$ is not fulfilled [13]. In particular, the escape events are no longer governed by an exponential decay law. Equivalently, the limits $D \to 0$ and $\tau \to \infty$ do not commute [43]. Exceptions are type II potentials $W(x)$ that satisfy (4.24) and (4.25). Within the validity of the rate concept we mainly restricted ourselves to the exponentially leading weak-noise contribution $\Delta\phi(\tau)$ to the mean escape time

(5.8). It is only for small and large τ that also the preexponential factor $\zeta(\tau, D)$ from (5.9) can be readily determined indirectly by means of (3.7) and (4.11), (5.10), respectively. For more general τ values, the determination of $\zeta(\tau, D)$ is a very difficult unsolved problem [34]. However, our indirect results for small and large τ and the general theory elaborated in Refs. [33,34,44] suggest that the function $\zeta(\tau, D)$ converges to a finite limit for $D \to 0$. Under the additional assumption that the τ dependence of this limiting function $\zeta(\tau, 0)$ is sufficiently smooth, it follows that $\Delta\phi(\tau)$ describes not only the dominating weak-noise behavior of $\bar{T}(\tau)$ for any fixed τ but also the qualitative features (monotonicity, extrema) of the τ dependence. Put differently, differentiation with respect to τ commutes with the limit $D \to 0$ in (5.8).

For potentials $W(x)$ of type I we proved that $\Delta\phi(\tau)$ is strictly monotonically decreasing with τ. Together with the kinetic model this implies that for sufficiently weak noise D the mean escape time $\bar{T}(\tau)$ first decreases from $\bar{T}(0)$ given in (3.8) [see also (3.1), (3.5)] towards $\bar{T}_0$ from (4.7) and then increases again towards $\bar{T}(\infty)$ from (4.9) [see also (3.3), (3.5)]. The minimum resonant activation $\tau_{\rm RA}$ occurs in the rather extended regime $\ln 1/D \ll \tau \ll \bar{T}(\tau)$ where both the rate concept and the kinetic model provide very good approximations and thus $\bar{T}(\tau)$ is almost constant, see Eq. (5.10). The breakdown of the rate concept is thus crucial for the occurrence of resonant activation of this type. In particular, $\tau_{\rm RA}$ diverges in the weak-noise limit $D \to 0$. So, for potentials $W(x)$ of type I the essential qualitative and quantitative properties of the mean escape time $\bar{T}(\tau)$ are understood quite well for weak noise D, see Fig. 1. A corresponding typical successful escape attempt of a particle (2.1) is sketched in Fig. 5.

For small τ a series expansion of $\Delta\phi(\tau)$ is straightforward. We left out this calculation here since the full mean escape time $\bar{T}(\tau)$ is known for small τ anyway, see (3.1), (3.5). For completeness, we only mention that in leading order τ one recovers the exponentially leading contributions in (3.7) for $\Delta\phi(\tau)$ and (7.14) for the most probable escape path [46].

If the fluctuating part of the potential, i.e., γ in (7.1), is small then $\Delta\phi(\tau)$ takes the form (7.6) for arbitrary τ. Note that even for small γ the variations of $\bar{T}(\tau) \simeq e^{\Delta\phi(\tau)/D}$ as a function of τ become exponentially large for weak noise D. The same stays true even if $W(x)$ is no longer considered as D independent but decreases with D slower than $\sqrt{D}$ [48].

When $W'(0) = W'(1) = 0$ and $U(x) - \Delta W\, W(x)$ is strictly monotonically increasing for $0 < x < 1$ then the large-τ asymptotics (8.11) for $\Delta\phi(\tau)$ is valid. The derivation of this result is rather involved since a series expansion in powers of $1/\tau$ about $\tau = \infty$ is not possible, in general. With respect to $\bar{T}(\tau)$ we are thus dealing with a "doubly singular" perturbation theory about $D = 0$ and $\tau = \infty$.

From these results for $\Delta\phi(\tau)$ and the kinetic model it follows that for potentials of type II and weak noise D the mean escape time $\bar{T}(\tau)$ decreases for small τ, displays a minimum in the region $\tau = O(1/\Delta U)$, and increases beyond this region, see Figs. 2 and 4. In addition to this absolute minimum at $\tau_{\rm RA}$ we cannot exclude the existence of further local minima in the domain $\tau = O(1/\Delta U)$. The latter actually seems likely for suitably chosen potentials $W(x)$ of type II with several extrema on $[0, 1]$. As in the type I case, the minimum of $\bar{T}(\tau)$ about $\tau_{\rm RA}$ is already present in the exponentially leading Arrhenius factor $e^{\Delta\phi(\tau)/D}$. But in contrast to the type I case, $\tau_{\rm RA}$ does not diverge if D approaches zero and the minimum of $\bar{T}(\tau)$ about $\tau_{\rm RA}$ becomes increasingly sharp. Moreover, the breakdown of the rate concept is not essential for the existence of resonant activation and may not even occur for $D \to 0$ and appropriate potentials $W(x)$. Figure 7 shows a typical successful escape attempt for potential of type II with a single hump in the interval $[0, 1]$ [49].

If $W(x)$ is of mixed type then for weak noise D the mean escape time is still monotonically decreasing for small τ, increasing towards $\bar{T}(\infty) > \bar{T}(0)$ within the validity of the kinetic model, and governed by (8.11) within the validity of this result. We thus expect that $\bar{T}(\tau)$ either shows the same qualitative features as in the type I or II case, or it is a "true mixture" with one or several minima in the region $\tau = O(1/\Delta U)$ and a further minimum in the domain $\ln 1/D \ll \tau \ll \bar{T}(\tau)$. In other words, we achieved a complete classification of the mean escape time $\bar{T}(\tau)$ for all possible potentials $W(x)$. Figures 1–3 and the example (7.12) considered in Sec. VII and Fig. 6 show that all three possible cases are actually realized. Moreover, we can predict that $\bar{T}(\tau)$ is type-II-like if the quantity E from (8.11) is negative and type-I-like or "truly mixed" otherwise. Note that these three different types of $\bar{T}(\tau)$ can be clearly distinguished only for weak noise D.

The theoretical and numerical results plotted in Figs. 1–4 display the correct tendency towards the predictions that the rate and kinetic descriptions approximately agree in a rather extended τ regime, $\bar{T}(\tau)$ is almost constant in this domain, and $\bar{T}(\tau)$ has a sharply peaked minimum in the region $\tau = O(1/\Delta U)$ for the examples shown in Figs. 2–4. For a more convincing illustration one should, however, consider even smaller noise strengths D.

Omitting the thermal noise $\xi(t)$ in (2.1) one recovers the common escape problem for a particle subject to colored noise only [50] except that the coupling of the white noise $\eta(t)$ in (2.3) is usually chosen not proportional to $\tau^{-1/2}$ but to τ^{-1}. Clearly, we can restrict ourselves to potentials $W(x)$ with a nonvanishing derivative in the region $0 \le x \le 1$ (i.e., a subclass of type I) since otherwise a particle (2.1) starting at the potential well $x = 0$ can never escape. Since $y(t)$ becomes white noise of vanishing intensity $\int_{-\infty}^{\infty} \langle y(t)\, y(0)\rangle dt$ for $\tau \to 0$ it is obvious that $\bar{T}(0) = \infty$. Similarly, in the static limit $\tau \to \infty$ one readily sees that $\bar{T}(\infty) = \infty$. However, for finite τ the mean escape time is finite as well. We thus recover a very pronounced form of resonant activation even in the absence of the thermal noise $\xi(t)$ that is only modified and actually diminished by including $\xi(t)$. From this point of view, resonant activation for this subclass of type I potentials $W(x)$ is not the effect of an interplay between thermal and potential fluctuations but rather a property of the escape problem with colored noise only that survives in the presence of additional white noise [11]. In contrast, for all other $W(x)$, in particular of type II and mixed type, resonant activation requires the interplay of both kinds of fluctuations. Returning to the case $W'(x) \neq 0$ for $0 \leq x \leq 1$ it is clear that omitting the thermal noise $\xi(t)$ in (2.1) is equivalent to an appropriate small-D and large-$W(x)$ limit. Consequently, for $\xi(t) = 0$ our result that the exponentially leading part (5.8) of $\bar{T}(\tau)$ decreases monotonically with τ stays true within the validity of the rate concept. However, the kinetic model for large τ introduced in Sec. IV A must be replaced by a different approach [51] since the assumption of a quasistatic potential during a successful escape attempt is no longer valid.

We close with an outlook regarding the full escape problem (2.1). One expects that this problem will, in general, be at least as hard to solve as the notorious escape problem with colored noise alone [50] since the latter follows by an appropriate large-$W(x)$ limit from the former [8]. It is only in the opposite limit of small $W(x)$ that a solution for the exponentially leading part $\Delta\phi(\tau)$ could be obtained relatively easily in the entire τ regime.

As far as small noise strengths D are concerned, there remains essentially one unsolved problem, namely a quantitative approximation for $\bar{T}(\tau)$ for intermediate correlation times $\tau = O(1/\Delta U)$, in particular for potentials $W(x)$ of type II and mixed type. Here, a promising approach seems to be the unified colored noise approximation (UCNA) [7,8,52]. In principle, it should also be possible to determine higher order γ contributions in (7.6) yielding accurate approximations for $\Delta\phi(\tau)$ even if $W(x)$ is not so small. Furthermore, it should be possible to derive exact expressions for $\Delta\phi(\tau)$ from (6.2) for simple specific potentials $U(x)$ and $W(x)$. A striking example for the value of such results for $\Delta\phi(\tau)$ is our prediction based on (7.6) that there must exist potentials $W(x)$ with $\bar{T}(\tau)$ showing a truly mixed type of behavior, see Figs. 3 and 6 and the discussion below Eq. (7.12). On the other hand, the examples from Fig. 4 show that a quantitative approximation for the prefactor $\zeta(\tau, D)$ in (5.9) may often be more important than an approximation for $\Delta\phi(\tau)$ going beyond (7.6).

If the noise strength D is no longer small, nothing is known except the small- and large-τ asymptotics of $\bar{T}(\tau)$ and hence the following existence criterion for resonant activation. In particular, both the rate concept and the kinetic model fail and one has to deal with the full master equation (3.10).

The consequences of our results for molecular motors [18] and selective pumps [17] are currently under investigation.

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APPENDIX A

We consider a potential $W(x)$ of type I. Without loss of generality we assume that $W'(x) \geq 0$ for $x \in [a, b]$ with $a \leq 0$, $b \geq 1$, and $W'(a) = W'(b) = 0$. We first show that $p(t) > 0$ for any finite t by means of an indirect proof. We start by assuming $p(t) = 0$ for a finite t. According to Hamilton's equation (6.5) this means $p(t) = 0$ for any t. With (6.9), (6.11), (6.12) it then follows that $q(t) - r(t) = y(t) = 0$ for any t. Since it is impossible to find a solution of Hamilton's equation (6.4) for $p(t) = y(t) = 0$ that satisfies the boundary conditions $x(t = -\infty) = 0$, $x(t = \infty) = 1$ our assumption $p(t) = 0$ must be wrong. Let us now assume that $p(t) < 0$ for a finite t. Again, this yields $p(t) < 0$ for any finite t due to Hamilton's equation (6.5). Next we show that this property $p(t) < 0$ implies $q(t) \geq 0$ for all sufficiently large t again by means of an indirect proof: If there are arbitrary large t values with $q(t) < 0$ then the boundary condition $x(t) \to 1$ for $t \to \infty$ and (6.11) yield $b = 1$ [i.e., $W'(1) = 0$] and $x(t) > 1$ for certain arbitrary large t values. The property $W'(1) = 0$ implies that the basin boundary δG_τ is given by the straight line $\{1\} \times \mathbb{R}$. Consequently, the MPEP $x(t)$, $y(t)$ crosses the basin boundary at a certain finite time t_0. Let us denote by $\bar{x}(t)$, $\bar{y}(t)$ a path that agrees with the MPEP for $t \leq t_0$ but deterministically approaches the saddle $(1, 0)$ for $t > t_0$ along the basin boundary (the nondifferentiability at $t = t_0$ does not play a role). With (5.7) we thus find that $L[x(t), y(t)] = L[\bar{x}(t), \bar{y}(t)]$ for $t \leq t_0$ and $L[x(t), y(t)] > L[\bar{x}(t), \bar{y}(t)] = 0$ for $t < t_0$ in contradiction to the fact that the MPEP $x(t), y(t)$ minimizes (6.2). Hence our assumption that $q(t) < 0$ must be wrong. Next, by exploiting $p(t) < 0$ and $q(t) \geq 0$ for sufficiently large t values we can rewrite (6.4) under the differential form $dx = dx_{\text{det}} + \delta x$, where dx_{det} is the differential flow of the deterministic dynamics (2.1) and $\delta x < 0$. Similarly, (6.6) is equivalent to $dy = dy_{\text{det}} + \delta y$, where dy_{det} accounts for the deterministic flow (2.3) and $\delta y \geq 0$. It is not difficult to see that under these conditions the MPEP $x(t)$, $y(t)$ cannot reach the point $(1, 0)$ for $t \to \infty$ without crossing the basin boundary δG_τ. [Note that close to the saddle $(1, 0)$ δG_τ is given in leading order approximation by $\tau W'(1) y = x - 1$ with $W'(1) \geq 0$.] This leads to an inconsistency by the same line of reasoning as before and thus our assumption $p(t) < 0$ was wrong.

We finally show that $W'(x(t)) \geq 0$ for all t by exploiting $p(t) > 0$. Let us assume that $x(t)$ leaves the domain with $W'(x) \geq 0$ in the negative direction for the first time at $t = t_0$, i.e., $x(t_0) = a \leq 0$, $\dot{x}(t_0) < 0$, and $W'(x(t_0)) = 0$. Since $p(t_0) > 0$ and $U'(a) \leq 0$ Hamilton's equation (6.4) yields the inconsistency $\dot{x}(t_0) > 0$ [remember that $U'(x) < 0$ for $x < 0$ and $x > 1$, see above Eq. (2.5)]. Thus $x(t) \geq a$ for all t. Similarly, one can show that the assumption $x(t_0) = b$ for some (finite) t_0 implies $\dot{x}(t_0) > 0$ and thus $x(t)$ never can return into the domain $x \leq b$ once it has left it. In view of the boundary condition $x(t = \infty) = 1 \leq b$ we thus can infer that $x(t) \leq b$ for all t.

APPENDIX B

In this appendix, we derive Eq. (7.6). With $x(t) = x_0(t) + \gamma\, x_1(t) + \cdots$, $y(t) = y_0(t) + \gamma\, y_1(t) + \cdots$ and taking into account (7.1), (7.2), and $x_1(t) = y_0(t) = 0$ the Lagrangian (5.7) takes the form

$$L = \tfrac{1}{4}\left[2\,U'(x_0) + \gamma^2\left\{\dot{x}_2 + x_2\,U''(x_0) + y_1\,W'(x_0)\right\}\right]^2 + \frac{\gamma^2\tau}{4}\left[\dot{y}_1 + \frac{y_1}{\tau}\right]^2 , \tag{B1}$$

where we omitted arguments t and contributions higher than of order $O(\gamma^2)$. Collecting terms of order γ^1 in (6.6), (6.7) one readily sees that

$$[\dot{y}_1 + y_1/\tau]^2 = [2\,q_1/\tau]^2 = 4\,q_1\,[\dot{q}_1 - W'(x_0)\,p_0]/\tau\ . \tag{B2}$$

Introducing (B2) into (B1) and using once more (7.2) one obtains

$$\begin{aligned} L &= \dot{x}_0\,U'(x_0) + \gamma^2[\dot{x}_2\,U'(x_0) + x_2\,\dot{x}_0\,U''(x_0) \\ &\quad + \dot{q}_1\,q_1 + \dot{x}_0\,W'(x_0)\,(y_1 - q_1)] \\ &= \frac{d}{dt}\left[U(x_0) + \gamma^2\left(x_2\,U'(x_0) + \frac{q_1^2}{2}\right)\right] \\ &\quad -\gamma^2\,\dot{x}_0\,W'(x_0)\,r_1\ . \end{aligned} \tag{B3}$$

Since $x_2(t)$ and $q_1(t)$ vanish for $t = \pm\infty$ the total derivative in (B3) contributes ΔU to the integral in (6.2). Regarding the last term in (B3) one first goes over from the integration over t in (6.2) to an integration over x_0 and then introduces the result (7.4) for $r_1(x_0)$. Finally, one recovers (7.6) by taking into account that there can be no terms of order $O(\gamma^3)$ due to the symmetry of $\Delta\phi(\tau)$ under $\gamma \mapsto -\gamma$ following from (6.2) and (5.7).

APPENDIX C

In this appendix, Eq. (7.9) is derived. For any fixed x, y, τ a Taylor expansion of $e^{-\Delta/\tau}$, $\Delta := \int_x^y dz/U'(z)$, yields the exact identity

$$e^{-\Delta/\tau} = 1 - \frac{\Delta}{\tau} + \frac{1}{2}\left(\frac{\Delta}{\tau}\right)^2 e^{-\theta\,\Delta/\tau} \tag{C1}$$

for an appropriate $\theta \in [0,1]$ depending upon x, y, and τ. Neglecting the last term in (C1) we thus can rewrite (7.6) as

$$I(\tau) = \int_0^1 dx \int_x^1 dy\, W'(x)\, W'(y)\left(1 - \frac{1}{\tau}\int_x^y \frac{dz}{U'(z)}\right)\ . \tag{C2}$$

For the first summand on the right-hand side one readily obtains $\Delta W^2/2$. Regarding the second term, we replace $W'(y)$ by $d[W(y) - W(1)]/dy$. Then a partial integration with respect to y yields

$$I(\tau) = \frac{\Delta W^2}{2} + \frac{1}{\tau}\int_0^1 dx\, W'(x)\int_x^1 dy\,\frac{W(y) - W(1)}{U'(y)}\ . \tag{C3}$$

After exchanging the order of integrations in the last term the x integral can be performed and one obtains (7.9) provided our approximation in (C1) is justified. In other words, we still have to prove that

$$\int_0^1 dx \int_x^1 dy\, W'(x)\, W'(y)\,\frac{1}{2}\left(\frac{\Delta}{\tau}\right)^2 e^{-\theta\,\Delta/\tau} = O(\tau^{-2})\ . \tag{C4}$$

Since $W'(x)$ is bounded and $0 \le e^{-\theta\,\Delta/\tau} \le 1$ for any τ and $x \le y$, it is sufficient to prove that

$$\int_0^1 dx \int_x^1 dy\left[\int_x^y \frac{dz}{U'(z)}\right]^2 < \infty\ . \tag{C5}$$

Since $U(x)$ has quadratic extrema at $x = 0$ and $x = 1$ and is strictly monotonically increasing in between, the integrand $\int_x^y dz/U'(z)$ is finite except for logarithmic singularities when $y \to 1$ or $x \to 0$ and (C5) follows.

APPENDIX D

In this appendix, the properties of the MPEP for single humped potentials $W(x)$ of type II stated below (7.13) are proven. Since $W'(x) \le 0$ for $\bar{x} \le x \le 1$ it is obvious from (7.3) that $q_1(x_0) \ge 0$ for $\bar{x} \le x_0 \le 1$. Moreover, since $W(x)$ is nontrivial, $q_1(\bar{x})$ is actually positive. On the other hand, $q_1(x_0)$ must be negative for sufficiently small $x_0 > 0$ since $\int_0^1 W'(y)\,dy = \Delta W = 0$ and the kernel $K(y, x_0)$ in (7.3) favors small y values and thus positive $W'(y)$ values. Next we note that in order γ^1 and using x_0 instead of t as parameter Eqs. (6.6), (6.7) take the form

$$y_1'(x_0) = \frac{2\,q_1(x_0) - y_1(x_0)}{\tau\,U'(x_0)}, \tag{D1}$$

$$q_1'(x_0) = W'(x_0) + \frac{q_1(x_0)}{\tau\,U'(x_0)}\ . \tag{D2}$$

Since $q_1(x_0) < 0$ for small x_0, $W'(x_0) \ge 0$ for $0 \le x_0 \le \bar{x}$, and $q_1(\bar{x}) > 0$ we can infer from (D2) that in the region $0 \le x_0 \le \bar{x}$ the function $q_1(x_0)$ switches from negative to positive values but once it is positive it stays positive as long as $x_0 < 1$. With $q_1(x_0) \ge 0$ for $\bar{x} \le x_0 \le 1$ it follows that $q_1(x_0)$ changes sign exactly once on the interval $[0,1]$. Similar properties as for $q_1(x_0)$ can be proven for $r_1(x_0)$ showing that $y_1(x_0) = q_1(x_0) - r_1(x_0)$ is negative for sufficiently small $x_0 > 0$ and positive for sufficiently small $(1 - x_0) > 0$. From (D1) it follows that $y_1(x_0)$ stays negative at least as long as $q_1(x_0)$ is negative. In the domain where $q_1(x_0)$ is positive $y_1(x_0)$ switches from negative to positive values but once it is positive it stays positive as long as $x_0 < 1$.

APPENDIX E

In this appendix, Eqs. (8.7)–(8.9) are derived. According to (8.2) we have $\bar{V}'(x) > 0$ for $0 < x < 1$

and $\bar{V}'(0) = 0$. Further, if one neglects the term $2\delta p(x)$ in (8.3) it is obvious that there exist positive constants A_1, A_2 with $A_1 x < \bar{V}'(x) < A_2 x$ for small x since $W'(0) = 0$, $y(x)$ is bounded and approaches 0 for $x \to 0$, and $U(x)$ has a quadratic minimum at $x = 0$. The same is true for large τ even if one includes the term $2\,\delta p(x)$ for the following reason: For small x values $\bar{V}'(x)$ and thus $\delta p(x)$ cannot decrease slower than proportional to x since otherwise one could infer from (8.2) that $x(t) = 0$ for a certain $t > -\infty$. In combination with the assumption (ii) that $\delta p(x) \to 0$ for $\tau \to \infty$ this shows that $2\,\delta p(x)$ is negligible in comparison with $U'(x) + y(x)\,W'(x)$ for sufficiently small x and large τ. Similarly, one can show that there exist positive constants B_1, B_2 with $B_1(1-x) < \bar{V}'(x) < B_2(1-x)$ for $x \to 1$ and large τ. Consequently, $\delta p(x)/\bar{V}'(x)$ is bounded and

$$\bar{\Delta} := \int_x^y \frac{dz}{\bar{V}'(z)} \tag{E1}$$

diverges logarithmically for $x \to 0$ or $y \to 1$ and is finite and positive otherwise $(0 \le x \le y \le 1)$.

Similarly as in Appendix C we rewrite the kernel (8.6) by means of a Taylor expansion as

$$e^{-\bar{\Delta}/\tau} = 1 - \frac{\bar{\Delta}}{\tau} + \frac{1}{2}\left(\frac{\bar{\Delta}}{\tau}\right)^2 e^{-\bar{\theta}\,\bar{\Delta}/\tau} \tag{E2}$$

where $\bar{\theta} \in [0,1]$ for any x, y, and τ. Next we estimate the contribution of the last term in (E2) to the integral (8.4). Taking into account that $0 \le e^{-\bar{\theta}\,\bar{\Delta}/\tau} \le 1$ for any $\tau \ge 0$ and $0 \le x \le y \le 1$, $\bar{\Delta}$ diverges logarithmically for $x \to 0$ or $y \to 1$, $W'(y)$ and $\delta p(y)/\bar{V}'(y)$ are bounded, and $W'(1) = 0$ there must exist constants $\bar{\alpha}$ and $\bar{\beta}$ such that

$$\left|\int_x^1 W'(y)\left(1 - \frac{\delta p(y)}{\bar{V}'(y)}\right)\frac{1}{2}\left(\frac{\bar{\Delta}}{\tau}\right)^2 e^{-\bar{\theta}\,\bar{\Delta}/\tau} dy\right| \le \frac{\bar{\alpha} + \bar{\beta}\,[\ln x]^2}{\tau^2}\ . \tag{E3}$$

The right-hand side is of the order $o(1/\tau^{2-\epsilon})$ whenever $x \ge x_\tau$, $x_\tau := \exp\{-\tau^{\epsilon/3}\}$, for any small $\epsilon > 0$. Consequently, Eq. (8.4) can be rewritten under the form

$$\begin{aligned} q(x) = {} & W(x) - W(1) + \frac{1}{\tau}\int_x^1 dy\, W'(y) \int_y^x \frac{dz}{\bar{V}'(z)} \\ & + \int_x^1 dy\, \frac{W'(y)\,\delta p(y)}{\bar{V}'(y)} \left\{1 - \frac{1}{\tau}\int_y^x \frac{dz}{\bar{V}'(z)}\right\} \\ & + o(1/\tau^{2-\epsilon}) \end{aligned} \tag{E4}$$

for $x \ge x_\tau$. In the first integral, the integration over y can be readily performed after exchanging the order of the integrations. In the second integral, the curly brackets can be rewritten as $1 + o(1/\tau^{1-\epsilon})$ by similar arguments as in (E3).

Next we introduce Hamilton's equation (6.4) and the definitions (6.9), (8.1) into (6.8) to yield

$$\delta p(x) = -\frac{q(x)\,r(x)}{\tau\, p(x)}\ . \tag{E5}$$

Since $p(x) = \bar{V}'(x) - \delta p(x)$ [see (8.1) and (8.2)], $\delta p(x)$ becomes a small quantity for $\tau \to \infty$ [assumption (ii)], and $r(x)$ is bounded [assumption (iii)] it follows that Eqs. (E4) and (E5) can be recast into (8.8) and

$$\delta p(x) = -\frac{r(x)\,[W(x) - W(1)]}{\tau\,\bar{V}'(x)} + O(1/\tau^2) \tag{E6}$$

for $x \ge x_\tau$. For $0 \le x \le x_\tau$ we can rewrite (8.4) as

$$\begin{aligned} q(x) = {} & q(x_\tau)\,\bar{K}(x_\tau, x) \\ & - \int_x^{x_\tau} W'(y)\left[1 - \frac{\delta p(y)}{\bar{V}'(y)}\right] \bar{K}(y,x)\, dy\ . \end{aligned} \tag{E7}$$

The term $\bar{K}(x_\tau, x)$ on the right-hand side monotonically decreases towards zero for $x \to 0$, first very slowly as long as x is still comparable to x_τ, but very rapidly for $x \ll x_\tau$. The last term in (E7) is zero for $x = x_\tau$ and $x = 0$ and at most of order $O(x_\tau)$ otherwise. Thus $q(x)$ is bounded, $|q(x)| \le |q(x_\tau)| + O(x_\tau)$, in the region $0 \le x \le x_\tau$ and goes to zero rapidly for $x \ll x_\tau$. A similar line of reasoning yields (8.9) for $x \le 1 - x_\tau$ and in the domain $1 - x_\tau \le x \le 1$ one finds that $|r(x)| \le |r(1-x_\tau)| + O(x_\tau)$ and $r(x) \to 0$ for $x \to 1$. Introducing (8.9) into (E6) one recovers (8.7) at least for $x_\tau \le x \le 1 - x_\tau$. From (E5) and our results for $q(x)$ and $r(x)$ it follows that the expression (8.7) overestimates the true $\delta p(x)$ for $0 \le x \le x_\tau$. However, since (8.7) decreases at least proportionally to x in this domain, the difference with the true $\delta p(x)$ is at most of order $O(x_\tau)$. Because $O(x_\tau)$ is smaller than $O(1/\tau^2)$ anyway, Eq. (8.7) stays true for $0 \le x \le x_\tau$ and similarly for $1 - x_\tau \le x \le 1$.

APPENDIX F

In this appendix, Eq. (8.11) is derived. By means of Hamilton's equations (6.4)–(6.7) the Lagrangian (5.7) can be rewritten under the form

$$L = p\,[\dot{x} + U'(x) + y\,W'(x)]/2 + q\,[\dot{y} + y/\tau]/2\ , \tag{F1}$$

where arguments t are omitted. Using again (6.7) the term $q\,[\dot{y} + y/\tau]$ can be recast into $d[q\,y]/dt - p\,y\,W'(x)$. Together with (8.1) this yields

$$L = (\dot{x} - \delta p)\,[p + \delta p + U'(x)]/2 + d[q\,y/2]/dt\ . \tag{F2}$$

Introducing (8.1) into Hamilton's equation (6.4) one obtains

$$p = U'(x) - \Delta W\,W(x) + \delta p + \delta y\,W'(x)\ , \tag{F3}$$

$$\delta y := y + \Delta W. \tag{F4}$$

Eliminating p in (F2) by means of (F3) implies

$$L = \frac{d}{dt}\left(U(x) - \frac{\Delta W\, W(x)}{2} + \frac{q\,y}{2}\right) + \delta y\, \frac{\dot{x}\, W'(x)}{2} - \delta p\, \frac{\Delta W\, W'(x)}{2} + \delta y\, \delta p\, \frac{W'(x)}{2} + \delta p^2 \,. \quad \text{(F5)}$$

After substitution of (F5) into (6.2) and taking into account that $q(t)$ and $y(t)$ vanish for $t = \pm\infty$ one finally obtains the exact identity

$$\Delta\phi(\tau) = \Delta U - \Delta W^2/2 + R_1/2 - R_2/2, \quad \text{(F6)}$$

$$R_1 := \int_{-\infty}^{\infty} dt\, W'(x(t))\, \delta y(t)\, [\dot{x}(t) + \delta p(t)] \,, \quad \text{(F7)}$$

$$R_2 := \int_{-\infty}^{\infty} dt\, \delta p(t)\, [\Delta W\, W'(x(t)) - 2\, \delta p(t)] \,. \quad \text{(F8)}$$

By means of (8.2) we can rewrite R_1 as

$$R_1 = \int_0^1 dx\, W'(x)\, \delta y(x) \left[1 + \frac{\delta p(x)}{\bar{V}'(x)}\right] \,. \quad \text{(F9)}$$

For $\delta p(x)$ and $\delta y(x) = \Delta W + q(x) - r(x)$ we can use the approximations (8.7)–(8.9) [with $V'(x)$ instead of $\bar{V}'(x)$, see text below Eq. (8.10)]. Strictly speaking, this approximation for $\delta y(x)$ is valid only for $x_\tau \le x \le 1 - x_\tau$ but one can readily see that it may be extended to the entire interval $[0,1]$ without changing the accuracy $o(1/\tau^{2-\epsilon})$ of the result for R_1 in (F9). Likewise, the term $[1+\delta p(x)/\bar{V}'(x)]$ in (F9) may be approximated by 1 without changing this accuracy. In summary, we obtain

$$R_1 = \frac{1}{\tau}\int_0^1 W'(x)\left[\int_x^1 \frac{W(1) - W(y)}{V'(y)}\, dy + \int_0^x \frac{W(y) - W(0)}{V'(y)}\, dy + \int_0^1 \frac{W'(y)\,[W(1) - W(y)]\,[W(y) - W(0)]}{V'(y)^2}\, dy\right] dx + o(1/\tau^{2-\epsilon}) \,. \quad \text{(F10)}$$

After exchanging the order of integrations the x integral can be performed. The evaluation of R_2 from (F8) is possible by similar arguments. Introducing R_1 and R_2 into (F6) finally yields (8.11).

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[35] This motivates the choice of the coupling strength $\sqrt{2D/\tau}$ in (2.3). Other choices have been studied for instance in Refs. [8–10,16]. Depending on this choice, not only the qualitative features of the mean escape time $\bar{T}$ as a function of the correlation time τ may be different [7,11,12] but also the difficulties and appropriate methods concerning the small- and large-τ analysis [8].

[36] Without this condition a rate description of the escape problem (2.1) for weak thermal noise becomes problematic even without potential fluctuations, $y(t) = 0$, see Sec. 4 E in Ref. [21].

[37] This condition is introduced essentially in order to guarantee that no initial point (x, y) diverges deterministically to $x = -\infty$ and that expressions like (3.1) or (3.3) stay finite for any $x \le x_{\text{th}}$. It thus can be easily relaxed. For instance, $W(x)$ proportional to $(1-x)^2$, which is of particular interest in the context of dye lasers [8,12,16], turns out to be an admissible choice provided $U(x)$ diverges sufficiently fast for $x \to -\infty$.

[38] This definition of the mean escape time $\bar{T}$ guarantees that it is equal to the inverse escape rate, independently of the exact choice of x_{th} and $\rho_0(x)$, provided the rate concept [21] applies at all. An alternative quantity that is often considered in the literature would be the mean-first-passage time across the basin boundary δG_τ which equals *half* the inverse escape rate within the validity of the rate concept [21]. For our purposes, the concept of the mean escape time $\bar{T}$ is easier to handle than the mean-first-passage time.

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[41] Note that (4.24) implies a local extremum of $F(v, w)$ at $v = 1$, $w = 0$ but not necessarily (4.25), whereas (4.25) implies $\Delta W = 0$ or $W'(0) = W'(1) = 0$ but not necessarily (4.24).

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[46] It has recently been pointed out [8] that if the white noise $\eta(t)$ in (2.3) couples not to $\tau^{-1/2}$ but to τ^{-1} then the weak noise asymptotics of the mean escape time becomes a very difficult and still not completely solved problem. In our case of a $\tau^{-1/2}$-coupling things are much easier and our leading order asymptotics (3.1), (3.5) indeed agrees with all previously derived approximations [8,10,11,13,47].

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[48] In this case we are dealing strictly speaking with a prefactor effect; as before the customary part $\bar{T}(\tau)\, e^{-\Delta\phi(\tau)/D}$ of the prefactor is expected to become asymptotically D independent.

[49] Figure 7 is certainly correct if $U'(x)$ and $W(x)$ are symmetric about $x = 1/2$ for $0 \le x \le 1$ or if either τ or $W(x)$ is small. If these conditions are not met or if $W(x)$ is not single humped then a typical successful escape attempt may display qualitatively different features, as discussed at the end of Sec. VIII.

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